Toppersnotes
Unleash the topper in you

UP – TGT

प्रशिक्षित स्नातक शिक्षक

उत्तर प्रदेश माध्यमिक शिक्षा सेवा चयन बोर्ड

गणित

भाग – 3

SIERRA INNOVATIONS

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☆ समाकलन का मूलभूत प्रमेय ☆

(Fundamental thearem of Integral Calculas)

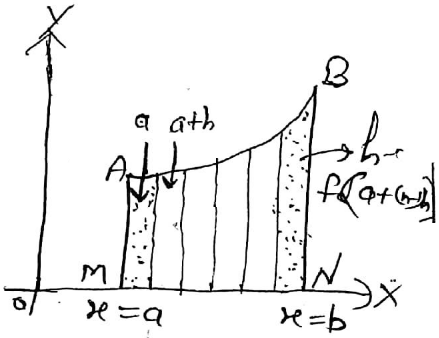
यदि $f(x)$ अंतराल $[a,b]$ में परिभाषित सतत फलन हो तब –

$$\int_a^b f(x)\,dx = \lim_{h\to\infty} h\left[f(a)+f(a+h)+f(a+2h)+\cdots\cdots+f(a+\overline{n-1}h)\right]$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \{h\cdot f(a+rh)\}$$

where $nh = b-a$

और $\boxed{\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{r=0}^{\infty} \frac{1}{n}\left(\frac{r}{n}\right)}$



Let $a = 0$
$b = 1$
then $nh = 1-0$
$h = \frac{1}{n}$

Similarly → $\frac{r}{n} = x$, $\frac{1}{n} = dx$

$\lim_{n\to\infty} \Sigma =$

Q.) सीमा $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\right)$ का मान ज्ञात करो।

∵ given सीमा → $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\left(\frac{1}{n}\right)}{\left(1+\frac{r}{n}\right)}$$

$$= \int_0^1 \frac{1}{1+x}\,dx = \left[\log(1+x)\right]_0^1$$

$$= \log 2 - \log 1 = \log 2$$

गहे व त्रि समाकल :⇒

Eg:- ① द्विसमाकल $\int_0^1 \int_{\sqrt{x}}^{x} (x^2+y^2)\,dx\,dy$ का मान ज्ञात करो।

Solⁿ— $\int_0^1 \left[\int_{\sqrt{x}}^{x} (x^2+y^2)\,dy \right] dx$

$\Rightarrow \int_0^1 \left(x^2 y + \frac{y^3}{3} \right)_{\sqrt{x}}^{x} dx$

$\Rightarrow \int_0^1 \left(x^3 + \frac{x^3}{3} - \sqrt{x}\cdot x^2 - \frac{x\sqrt{x}}{3} \right) dx$

$\Rightarrow \int_0^1 \left(x^3 + \frac{x^3}{3} - x^{5/2} - \frac{x^{3/2}}{3} \right) dx$

$\Rightarrow \left(\frac{x^4}{4} + \frac{x^4}{12} - \frac{2x^{7/2}}{7} - \frac{2}{5}\frac{x^{5/2}}{3} \right)_0^1$

$\Rightarrow \left(\frac{1}{4} + \frac{1}{12} - \frac{2}{7} - \frac{2}{15} \right) = \frac{4}{12} - \frac{4}{35}$

$\Rightarrow \frac{140-48}{12 \times 35}$

Q.194) $\int_0^1 \int_0^2 (x+y)\,dx\,dy$

$\Rightarrow \int_0^1 \left(xy + \frac{y^2}{2} \right)_0^2 dx$

$\Rightarrow \int_0^1 \left(2x + \frac{4}{2} \right) dx$

$\Rightarrow \left(\frac{2x^2}{2} + 2x \right)_0^1 \Rightarrow 1+2 = 3$

Q.192) $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-x^2}\cdot e^{-y^2}\,dy\,dx$$

$$\Rightarrow \int_0^\infty \left[\int_0^\infty e^{-x^2}\,dy\right] e^{-y^2}\,dy$$

$$\Rightarrow \int_0^\infty \frac{1}{2}\Gamma\left(\frac{1}{2}\right) e^{-y^2}\,dy$$

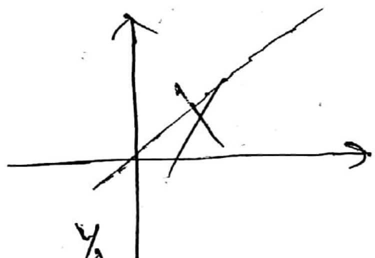
$$\Rightarrow \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\int_0^\infty e^{-y^2}\,dy$$

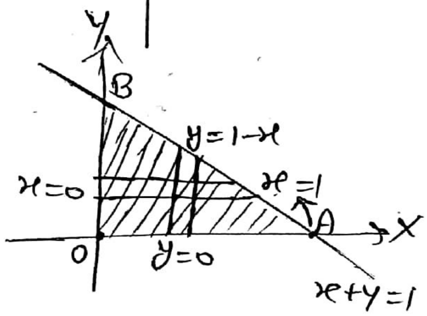
$$\Rightarrow \frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Rightarrow \frac{\pi}{4}$$

Q.182) $\iint xy\,dx\,dy$ $\qquad x+y \le 1$

~~दिये गये समाकलन क्षेत्र से —~~





$$\int_0^1\int_0^{1-x} xy\,dx\,dy$$

$$\Rightarrow \int_0^1 \left(\frac{xy^2}{2}\right)_0^{1-x} dx$$

$$\Rightarrow \int_0^1 \frac{x(1-x)^2}{2}\,dx$$

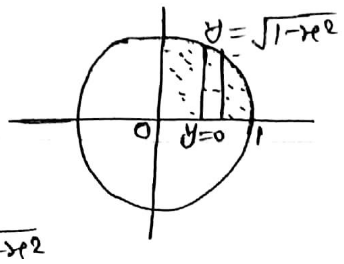
$$\Rightarrow \int_0^1 \frac{x(1-2x+x^2)}{2}\,dx$$

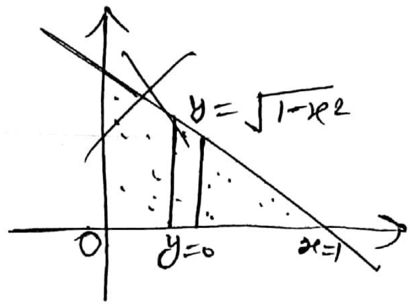
$$\Rightarrow \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx$$

$$\Rightarrow \left(\frac{x^2}{4} - \frac{x^3}{3} + \frac{x^4}{8}\right)_0^1$$

$$\Rightarrow \frac{1}{4} - \frac{1}{3} + \frac{1}{8} = \frac{6-8+3}{24} = \frac{1}{24}$$ Ans.

Q. 18) $\iint_R x^2y^2\,dx\,dy$ $\quad R = x \geq 0, \; y \geq 0 \quad x^2+y^2 \leq 1$





$$\Rightarrow 4\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2\,dx\,dy$$

$$\Rightarrow 4\int_0^1 \left(x^2\frac{y^3}{3}\right)_0^{\sqrt{1-x^2}}$$

$$\Rightarrow 4\int_0^1 \frac{x^2\sqrt{1-x^2}\,(1-x^2)}{3}\,dx$$

$$\Rightarrow 4\int_0^1 \frac{x^2(1-x^2)^{3/2}}{3}\,dx$$

$$\Rightarrow -\frac{4}{3}\times\frac{1}{2}\int_0^{\pi/2}\cos^2\theta \cdot (1-\cos^2\theta)^{3/2}\,(\sin\theta\, d\theta)$$ $\quad$ put $x = \cos\theta$, $dx = \sin\theta\, d\theta$

$$\Rightarrow \frac{1}{6}\int_0^{\pi/2}\cos^2\theta\,(\sin^2\theta)^{3/2}\sin\theta\,d\theta$$

$$\Rightarrow \frac{1}{6}\int_0^{\pi/2}\cos^2\theta\,\sin^4\theta\,d\theta$$

$$\Rightarrow \frac{1}{6}\,\frac{\Gamma\frac{3}{2}\,\Gamma\frac{5}{2}}{2\,\Gamma\frac{6+2}{2}} = \frac{1}{6}\,\frac{\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2\cdot 3\cdot 2\cdot 1}$$

$$= \frac{1}{6}\,\frac{\frac{3}{8}\pi}{\cancel{3}\,4} = \frac{1}{6}\times\frac{\cancel{4}\pi}{82}$$

$=$

वाली प्रमेय से - $\quad \frac{1}{3}\left[\frac{1\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}\right] = \frac{\pi}{96}$

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प्रश्न 8.) ~~द्विसमाकल~~ $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$ का क्रम बदलो।
याद

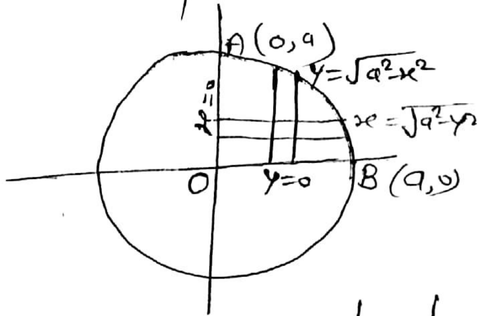
∵ द्विसमाकल का समाकलन क्षेत्र –

y का मान $y=0$ से $y=\sqrt{a^2-x^2}$ तक अर्थात

$y=0$ से $x^2+y^2=a^2$ तक है।

x का मान $x=0$ से $x=a$ तक है।

Now क्रम बदलने पर –



$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$

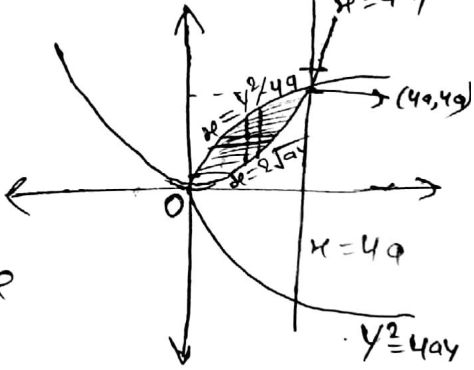
here x का मान समाकलन क्षेत्र के बांये वक्र से दांये वक्र तक होगा।

अर्थात $x=0$ से $x=\sqrt{a^2-y^2}$ तक रहेगा।

& y का मान $y=0$ से $y=a$ तक।

Imp. Q.
याद

$\int_0^{4a} \int_{\frac{x^2}{4a}}^{2\sqrt{ax}} f(x,y)\,dx\,dy$ का क्रम बदलो।



here $x=0$ to $x=4a$

$y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$

$x^2=4ay$ to $y^2=4ax$

क्रम बदलने पर –

$$\Rightarrow \int_0^{4a} \int_{2\sqrt{ay}}^{y^2/4a} f(x,y)\,dy\,dx$$

Q. 187) $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dx\, dy$ ILATE

$\Rightarrow \int_0^{\infty}\int_x^{\infty} [\underset{II}{e^{-y}} . \underset{I}{y^{-1}}]\, dx\, dy$

$\Rightarrow \int_0^{\infty} \left[-y^{-1} e^{-y} - \int \cancel{\log y . e^{y}} \frac{1}{y^2} e^{-y}\, dy\right]$

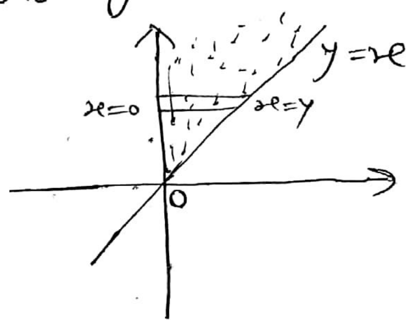
∵ ये fun. बढ़ता ही जा रहा है। अतः इसका y के resp. diff. firstly नहीं कर सकते

∴ हम इसका क्रम बदलेंगे।

∵ given limites $\Rightarrow$ $x = 0$ to $x = \infty$

$y = x$ to $y = \infty$

क्रम बदलने पर-



$\int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dy\, dx$

$\Rightarrow \int_0^{\infty} \frac{e^{-y}}{y} (x)_0^{y}$

$\Rightarrow \int_0^{\infty} \frac{e^{-y}}{\cancel{y}} \cancel{y}\, dy$

$\Rightarrow (-e^{-y})_0^{\infty} = -e^{-\infty} + e^{-0}$

$= 0 + 1 = 1$

Q. 188) $\int_0^1\int_0^{x^2} e^{y/x}\, dx\, dy$

$\Rightarrow \int_0^1 \left[\int_0^{x^2} e^{y/x}\, dy\right] dx$

$\Rightarrow \int_0^1 (x e^{y/x})_0^{x^2} dx \Rightarrow \int_0^1 (x e^{x} - x)\, dx$

$\Rightarrow (x e^{x})_0^1 - (e^{x})_0^1 - \left(\frac{x^2}{2}\right)_0^1$

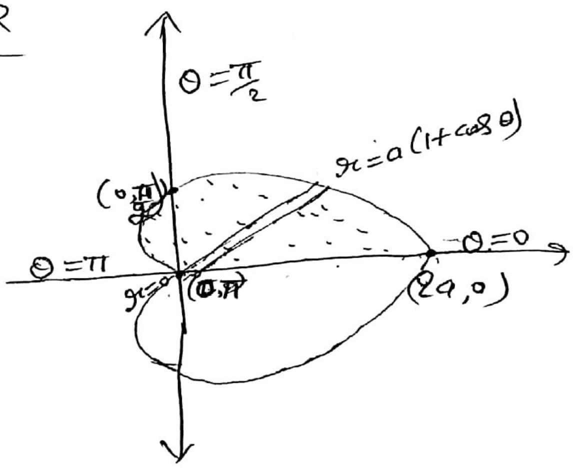
$\Rightarrow (e-0)-(e-1)-\left(\frac{1}{2}-0\right)$

$\Rightarrow \frac{1}{2}$

Q. 147) कार्डियाइड $r = a(1+\cos\theta)$ —(1)

$\because$ (1) का क्षे० $\Rightarrow \iint_R r\,d\theta\,dr$

$\because$ (1) में $\theta = -\theta$ put करने पर कोई अंतर नहीं है।
अतः Symmetric है।



$\theta = 0$ पर $r = 2a$

$\theta = \frac{\pi}{2}$ पर $r = a$

$\theta = \pi$ पर $r = 0$

$\Rightarrow \iint_R r\,d\theta\,dr$

$\Rightarrow 2$ [प्रारम्भिक रेखा के ऊपर वाले क्षेत्र का क्षेत्रफल]

$\Rightarrow 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,d\theta\,dr$

द्विसमाकल के कार्तीय रूप को ध्रुवीय रूप मे बदलना — $x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,d\theta\,dr$ put करेंगें।

Q.145) $\int_0^a\int_y^a \frac{x\,dy\,dx}{x^2+y^2}$

$x = y$ to $x = a$
$y = 0$ to $y = a$

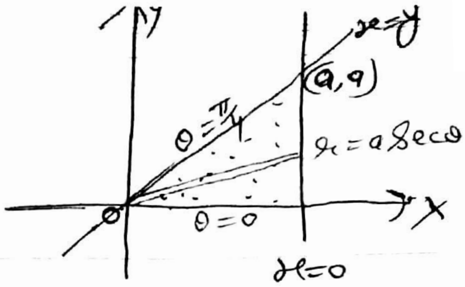
$\because x = r\cos\theta$

$\Rightarrow \iint \frac{\cancel{r}\cos\theta \cdot \cancel{r}\,d\theta\,dr}{\cancel{r^2}}$

$\cancel{\int\int \cos\theta \, d\theta \, dr}$

given समाकलन को ध्रुवीय रूप मे बदलने पर

$$\Rightarrow \int_0^{\pi/4}\int_0^{a\sec\theta} \cos\theta \, d\theta \, dr$$



$\because x = a \Rightarrow r\cos\theta = a$

$r = a\sec\theta$

$x = y \Rightarrow r\cos\theta = r\sin\theta$

$\Rightarrow \tan\theta = 1$

$\boxed{\theta = \frac{\pi}{4}}$

Q.2011 $\displaystyle\int_{-1}^{1}\int_0^{z}\int_{x-z}^{x+z} (x+y+z)\, dz\, dx\, dy$

$$\Rightarrow \int_{-1}^{1}\int_0^{z} \left(xy + \frac{y^2}{2} + yz\right)_{x-z}^{x+z} dz\, dx$$

$$\Rightarrow \int_{-1}^{1}\int_0^{z} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z) - x(x-z) - \frac{(x-z)^2}{2} - z(x-z)\right] dz\, dx$$

$$\Rightarrow \int_{-1}^{1}\int_0^{z} \left[x^2 + xz + \frac{x^2}{2} + \frac{2xz}{2} + \frac{z^2}{2} + zx + z^2 - \frac{x^2}{2} + xz - \frac{x^2}{2} + zx - z^2\right] dz\, dx$$

$$\Rightarrow \int_{-1}^{1}\int_0^{z} (x+z)^2 \, dz\, dx$$

$$\Rightarrow \int_{-1}^{1} \frac{(x+z)^3}{3}\, dz$$

$$\Rightarrow \int_{-1}^{1} \frac{(2z)^3}{3}\, dz$$

$$\Rightarrow \frac{8}{3}\left(\frac{z^4}{4}\right)_{-1}^{1} \Rightarrow \frac{2}{3}[1-1]$$

$$\Rightarrow 0$$

Q.2001 $\displaystyle\int_0^1\int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$

$$\Rightarrow \int_0^1 \left(x^2y + \frac{y^3}{3}\right)_x^{\sqrt{x}} dx$$

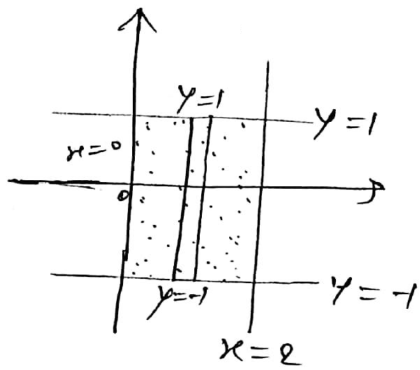
$$\Rightarrow \int_0 \left(x^{7/2} + \frac{x^{7/2}}{3} - x^3 - \frac{x^3}{3} \right) dx$$

$$\Rightarrow \int_0^1 \left(\frac{4}{3} x^{5/2} - \frac{4}{3} x^3 \right) dx$$

$$\Rightarrow \left(\frac{4}{3} \cdot \frac{2x^{7/2}}{7} - \frac{4}{3} \times \frac{x^4}{4} \right)_0^1$$

$$\Rightarrow \left(\frac{8}{35} - \frac{4}{3} \right) = \frac{8-5}{35} = \cancel{\frac{8}{15}}\ \frac{3}{35}$$

Q. 193) $\displaystyle\int_0^2 \int_{-1}^1 (1 - 6x^2 y)\, dx\, dy$



$$\Rightarrow \int_0^2 \left(y - \cancel{6}^{3} \frac{x^2 y^2}{\cancel{2}} \right)_{-1}^{1} dx$$

$$\Rightarrow \int_0^2 (y - 3x^2 y^2)_{-1}^{1}\, dx$$

$$\Rightarrow \int_0^2 \cancel{yx}\ (1 - \cancel{3x^2} + 1 + \cancel{3x^2})\, dx$$

$$\Rightarrow [2x]_0^2 \Rightarrow 4$$

Q. 177) $\displaystyle\int_0^a \int_0^{\sqrt{a^2 - x^2}} xy\, dx\, dy$

$$\Rightarrow \int_0^a x \left(\frac{y^2}{2} \right)_0^{\sqrt{a^2 - x^2}} dx$$

$$\Rightarrow \frac{1}{2} \int_0^a x (a^2 - x^2)\, dx$$

$$\Rightarrow \frac{1}{2} \int_0^a (x a^2 - x^3)\, dx$$

$$\Rightarrow \frac{1}{2} \left[a^2 \frac{x^2}{2} - \frac{x^4}{4} \right]_0^a$$

$$\Rightarrow \frac{1}{2} \left[\frac{a^4}{2} - \frac{a^4}{4} \right] \Rightarrow \frac{1}{2} \left(\frac{a^4}{4} \right) = \frac{a^4}{8}$$

Q. 178) $\displaystyle\int_0^{\pi} \int_0^{a \sin\theta} r\, dr\, d\theta$

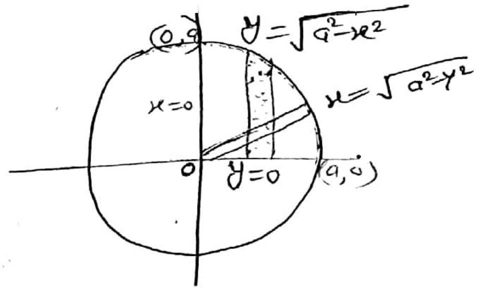
$$\Rightarrow \int_0^{\pi} \left(\frac{r^2}{2} \right)_0^{a \sin\theta} d\theta \Rightarrow a^2 \int_0^{\pi} \frac{\sin^2\theta}{2}\, d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0 \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$\Rightarrow \frac{a^2}{4}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi} \Rightarrow \frac{a^2}{4}\left[\pi - \frac{\sin 2\pi}{2}\right]$$

$$\Rightarrow \frac{a^2}{4}[\pi - 0] = \frac{a^2\pi}{4}$$

Q.181) $\int_0^a \int_0^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$



$x = 0$ to $x = a$

$y = 0$ to $y = \sqrt{a^2-x^2}$

$x^2+y^2 = a^2$

$$\Rightarrow \cancel{\int_0^{\sqrt{a^2-x^2}} \int_0^{a} f(x,y)\,dx\,dy}$$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dy\,dx$$

Q.175) $\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} (x^2+y^2+z^2)\,dx\,dy\,dz$

$$\Rightarrow \int_{-a}^{a}\int_{-b}^{b} \left(x^2z + y^2z + \frac{z^3}{3}\right)_{-c}^{c} dx\,dy$$

$$\Rightarrow \int_{-a}^{a}\int_{-b}^{b} \left(x^2c + y^2c + \frac{c^3}{3} + x^2c + y^2c + \frac{c^3}{3}\right) dx\,dy$$

$$\Rightarrow 2\int_{-a}^{a} \left[x^2cy + \frac{y^3}{3}c + \frac{yc^3}{3} + x^2cy + \frac{y^3}{3}c + \frac{c^3}{3}y\right]_{-b}^{b}$$

$$\Rightarrow 4\int_{-a}^{a} \left[x^2cb + \frac{b^3}{3}c + \frac{bc^3}{3}\right] dx$$

$$\Rightarrow 8\left[\frac{x^3}{3}cb + \frac{b^3}{3}cx + \frac{bc^3}{3}x\right]_0^a$$

$$\Rightarrow 8\left[\frac{a^3cb}{3} + \frac{ab^3c}{3} + \frac{abc^3}{3}\right] \Rightarrow \frac{8abc}{3}(a^2+b^2+c^2)$$

Q.19) $z = xy^2 + x^2y \quad ; \quad x = at^2, \; y = 2at$

$\because \dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial t}$

$= (y^2 + 2xy)(2at) + (2xy + x^2)(2a)$

$= 2aty^2 + 4xyat + 4xay + 2ax^2$

$= 2at(2at)^2 + 4(at^2)(2at)at + 4(at^2)a(2at) + 2a(at^2)^2$

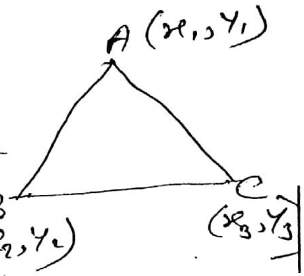
$= 8a^3t^3 + 8a^3t^4 + 8a^3t^3 + 2a^3t^2$

Q.21) $y^3 = 3ax^2 - x^3$

$3y^2\dfrac{dy}{dx} = 6ax - 3x^2$

$6y\dfrac{d^2y}{dx^2} = 6a - 6x$

Q.93)



$S = (x-x_1)^2 + (x-x_2)^2 + (x-x_3)^2 + (y-y_1)^2 + (y-y_2)^2 + (y-y_3)^2$

$p = 0,\; q = 0 \Rightarrow$

$x = \dfrac{x_1 + x_2 + x_3}{3}$

$y = \dfrac{y_1 + y_2 + y_3}{3}$

केन्द्रक

Q.104) $y = mx + am^p$

$y - mx - am^p = 0$ —— (1)

m के लिए diff. $\Rightarrow$

$0 - x - pam^{p-1} = 0$

$\Rightarrow m^{p-1} = -x/pa$

$\Rightarrow m = \left(\dfrac{-x}{pa}\right)^{\frac{1}{p-1}}$

(1) में m का मान put $\Rightarrow$

$y - \left(-\dfrac{x}{pa}\right)^{\frac{1}{p-1}}\cdot x - a\left(-\dfrac{x}{pa}\right)^{\frac{p}{p-1}} = 0$

$y = m(x - am^{p-1})$

$y = \left(\dfrac{-x}{pa}\right)^{\frac{1}{p-1}}(x + am^{p-1})$

$y = \left(\dfrac{-x}{ap}\right)^{\frac{1}{p-1}}\left[x + a\left(-\dfrac{x}{pa}\right)\right]$

$py = \left(\dfrac{-x}{ap}\right)^{\frac{1}{p-1}}\cdot x(p-1)$

$(py)^{p-1} = \dfrac{-x}{ap}\cdot x^{p-1}(p-1)^{p-1}$

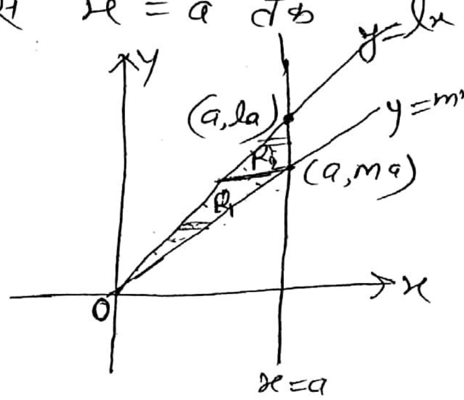
$\boxed{ap^py^{p-1} + x^p(p-1)^{p-1} = 0}$

Q.140) $\int_0^a \int_{mx}^{lx} V\, dx\, dy$ का क्रम बदलना है।

$\because$ given समाकल क्षेत्र $\Rightarrow$

y का मान $y = mx$ से $y = lx$ तक

$x = 0$ से $x = a$ तक



$$\Rightarrow \int_0^{ma} \int_{y/l}^{y/m} V\, dy\, dx + \int_{ma}^{la} \int_{y/l}^{a} V\, dy\, dx$$

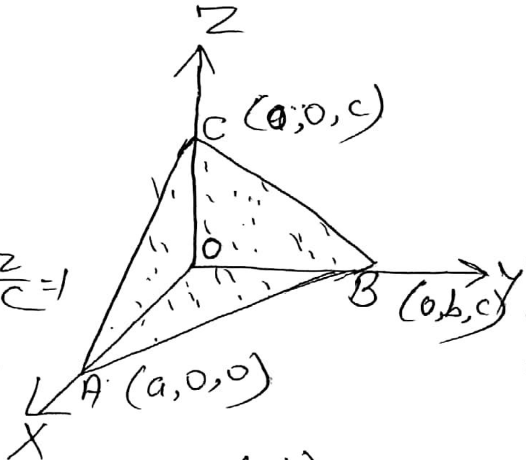
Q.148) $x = 0, y = 0, z = 0$

& plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

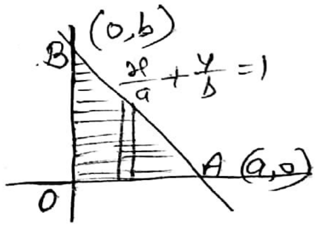
$OA = a$, $OB = b$, $OC = c$

समतल ABC का समी. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

दिये गये चारों समतलों से घिरे क्षे. का आयतन $\Rightarrow$



$$V = \iiint dx\, dy\, dz$$



$$= \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dx\, dy\, dz$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} [z]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dx\, dy$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\, dy$$

$$= \int_0^a c\left(y - \frac{xy}{a} - \frac{y^2}{2b}\right)_0^{b(1-\frac{x}{a})} dx$$

$$= \int_0^a c\left(b\left(1-\frac{x}{a}\right) - \frac{xb}{a}\left(1-\frac{x}{a}\right) - \frac{b}{2b}\left(1-\frac{x}{a}\right)^2\right) dx$$

$$= c\int_0^a \left(b - \frac{bx}{a} - \frac{xb}{a} + \frac{x^2 b}{a^2} - \frac{b^2}{2b} + \frac{\cancel{2}b^{\cancel{2}}x}{\cancel{2}a\cancel{b}} - \frac{x^2 b^{\cancel{2}}}{2a^2\cancel{b}}\right)$$

$$= c\int_0^a \left(b - \frac{2bx}{a} + \frac{x^2 b}{a^2} - \frac{b^2}{2b} + \frac{bx}{a} - \frac{x^2 b}{2a^2}\right) dx$$

$$= c\left[bx - \frac{\cancel{2}b}{a}\frac{x^2}{\cancel{2}} + \frac{x^3 b}{3a^2} - \frac{b^2 x}{2b} + \frac{bx^2}{2a} - \frac{x^3 b}{3a^2}\right)_0^a$$

$$= c\left[ab - \frac{ba^2}{a} + \frac{a^3 b}{3a^2} - \frac{b^2 a}{2b} + \frac{ba^2}{2a} - \frac{a^3 b}{3a^2}\right]$$

$$= c\left[\cancel{ab} - \cancel{ab} + \frac{ab}{3} - \cancel{\frac{ab}{2}} + \cancel{\frac{ab}{2}} - \frac{ab}{3}\right]$$

गामा फलन $\Rightarrow$ ① $\because$ $\Gamma(n+1) = n\Gamma(n)$

~~proof~~ $\Gamma(n+1) = \int_0^\infty x^n \cdot e^{-x}\, dx$
I II

$$= \left[x^n(-e^{-x})\right]_0^\infty - n\int_0^\infty x^{n-1}(-e^{-x})\,dx$$

$$= (0-0) + n\int_0^\infty x^{n-1}e^{-x}\,dx$$

$$\Rightarrow \boxed{\Gamma(n+1) = n\Gamma(n)}$$

② $\Gamma(n+1) = n\Gamma(n)$

$$= n\{(n-1)\Gamma(n-1)\}$$
$$= n(n-1)\Gamma(n-1)$$
$$= n(n-1)(n-2)\Gamma(n-2)$$
$$= n(n-1)(n-2)\ldots 2\cdot 1\cdot \Gamma(1)$$
$$= n(n-1)(n-2)\ldots 2\cdot 1\cdot 1$$
$$= n!$$

Let $x = \log\frac{1}{y}$

या $x = -\log y$

या $\log y = -x$

$\therefore y = e^{-x}$

$\therefore dy = e^{-x}(-dx)$

या $e^{-x}dx = -dy$

$\Gamma(n) = \int_1^0 \left(\log\frac{1}{y}\right)^{n-1}(-dy)$

$\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy$

या $\boxed{\Gamma(n) = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx}$

(V) $\Gamma(n) = \int x^{n-1}e^{-x}\,dx$

$$= \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\,dy$$

$$\boxed{n\Gamma(n) = \int_0^\infty e^{-y^{1/n}}\,dy}$$

$\left\{\because \text{let } x = y^{1/n} \Rightarrow x^n = y \Rightarrow nx^{n-1}dx = dy\right\}$

$n=\frac{1}{2}$ put करने पर

$$\int_0^\infty e^{-y^2}dy = \frac{1}{2}\sqrt{\tfrac{1}{2}}$$

या $$\boxed{\int_0^\infty e^{-x^2}dx = \frac{1}{2}\sqrt{\pi}}$$

(vii) $$\frac{\sqrt{n}}{a^n} = \int_0^\infty x^{n-1}e^{-ax}dx$$

$$= \int_0^\infty a^{n-1}y^{n-1}e^{-ay}\,a.dy$$

$$\Rightarrow \frac{\sqrt{n}}{a^n} = \int_0^\infty y^{n-1}e^{-ay}dy$$

y के स्थान पर x put -

$$\boxed{\frac{\sqrt{n}}{a^n} = \int_0^\infty x^{n-1}e^{-ax}dx}$$

(viii) $$\sqrt{n} = \int_0^\infty x^{n-1}e^{-x}dx$$

$\left\{ \text{Let } x=y^2,\ dx = 2y\,dy \right.$

$$\sqrt{n} = \int_0^\infty y^{2n-2}e^{-y^2}2y\,dy$$

$$= 2\int_0^\infty y^{2n-1}e^{-y^2}dy$$

$$\boxed{\sqrt{n} = 2\int_0^\infty x^{2n-1}e^{-x^2}dx}$$

β - फलन $\Rightarrow$ $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

~~$\sqrt{n} = \int_0$ prove~~

let $x = \frac{y}{y+1}$ ~~$= y(y+1)^{-1}$~~

~~$dx =$~~

$1-x = 1-\frac{y}{y+1} = \frac{1}{y+1}$

$$\Rightarrow dx = \frac{1}{(y+1)^2} dy$$

put in ① $\rightarrow$

$$B(m,n) = \int \left(\frac{y}{y+1}\right)^{m-1} \left(\frac{1}{y+1}\right)^{n-1} \frac{1}{(y+1)^2} dy$$

$$\Rightarrow \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$$\boxed{B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dy}$$

ii) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ — ①

Let $x = \sin^2\theta$
$dx = 2\sin\theta \cos\theta d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta \cos^{2n-2}\theta \cdot 2\sin\theta \cdot \cos\theta d\theta$$

$$\boxed{B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta}$$

$\because$ we know that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

by ① $\Rightarrow \int_0^1 (1-x)^{m-1} x^{n-1} dx$

$$\boxed{B(m,n) = B(n,m)}$$

iii) $\Gamma n \, \Gamma(1-n) = \frac{\pi}{\sin n\pi}$ $\quad 0 < n < 1$

put $n = \frac{1}{2}$

$\Gamma\frac{1}{2} \, \Gamma\frac{1}{2} = \frac{\pi}{\sin\frac{\pi}{2}} \Rightarrow \left(\Gamma\frac{1}{2}\right)^2 = \pi$

$$\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$