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CBSE

CLASS 12TH

THE CENTRAL BOARD OF SECONDARY EDUCATION

PART – VI

MATHEMATICS - I

SIERRA INNOVATIONS

MATHEMATICS - 1

CHAPTER-1 RELATIONS AND FUNCTIONS

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Introduction

The concept of the term 'relation' in mathematics has been drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is a recognisable connection or link between the two objects or quantities. Let A be the set of students of Class XII of a school and B be the set of students of Class XI of the same school. Examples of relations from A to B are
(i) $\{(a, b) \in A \times B$: a is brother of b$\}$,
If $(a, b) \in R$, we say that a is related to b under the relation R and we write as a R b. In general, $(a, b) \in R$, we do not bother whether there is a recognisable connection or link between a and b. As seen in Class XI, functions are special kind of relations.

Relation: A relation R from set X to a set Y is defined as a subset of the cartesian product $X \times Y$. We can also write it as $R \subseteq \{(x, y) \in X \times Y : xRy\}$.

Note: If $n(A) = p$ and $n(B) = q$ from set A to set B, then $n(A \times B) = pq$ and number of q

<u>**Types of Relations**</u>

Empty relation: A relation R in a set A is called empty relation, if no element of A is related to any element of A, i.e., $R = \phi \subset A \times A$.

Example: Consider the relation on the set $A = \{1,2,3,4,5\}$ defined by $R = \{(a,b): a - b = 12\}$.

Solution: Now , we observe that $a - b \neq 12$ for any two elements of A.R does not contain any element of A X A .

Hence, R is an empty relation.

Universal relation: A relation R in a set A is called universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$.
Both the empty relation and the universal relation are some times called trivial relations.

Example: Consider the relation on the set A ={1,2,3,4,5} defined by R ={(a,b): |a – b| ≥0}.

Solution: Now ,we observe that |a – b| ≥0 for all (a,b) € A.

(a,b) € R for all (a,b) € A X A.

R = A X A.

Hence, R is an universal relation on A.

Example: Let A be the set of all students of a boys school. Show that the relation R in A given by R = {(a, b) : a is sister of b} is the empty relation and R′ = {(a, b) : the difference between heights of a and b is less than 3 meters} is the universal relation.

Solution: Since the school is boys school, no student of the school can be sister of any student of the school. Hence, R = ϕ, showing that R is the empty relation. It is also obvious that the difference between heights of any two students of the school has to be less than 3 meters. This shows that R′ = A × A is the universal relation.

Conditions for checking reflexive, symmetric and transitive relations:

A relation R in a set A is called (i) reflexive, if (a, a) ∈ R, for every a ∈ A,
(ii) symmetric, if (a1 , a2) ∈ R implies that (a2 , a1) ∈ R, for all a1 , a2 ∈ A.
(iii) transitive, if (a1 , a2) ∈ R and (a2 , a3) ∈ R implies that (a1 , a3)∈ R, for all a1 , a2, a3 ∈A.

Reflexive Relation: A relation R defined on a set A is said to be reflexive, if
(x, x) ∈ R, ∀ x ∈ A or
xRx, ∀ x ∈ R

Example: A relation R is on set A (set of all integers) is defined by "x R y if and only if 2x + 3y is divisible by 5", for all x, y ∈ A. Check if R is a reflexive relation on A.

Solution: Let us consider x ∈ A.

Now 2x + 3x = 5x, which is divisible by 5.

Therefore, xRx holds for all ‘a’ in A

Hence, R is reflexive

Symmetric Relation: A relation R defined on a set A is said to be symmetric, if
$(x, y) \in R \Rightarrow (y, x) \in R, \forall x, y \in A$ or
$xRy \Rightarrow yRx, \forall x, y \in R$.

Example:Let R be a relation on Q, defined by $R = \{(a, b) : a, b \in Q \text{ and } a - b \in Z\}$. Show that R is Symmetric relation.

Solution:

Given $R = \{(a, b) : a, b \in Q, \text{ and } a - b \in Z\}$.

Let $ab \in R \Rightarrow (a - b) \in Z$, i.e. $(a - b)$ is an integer.

$\Rightarrow -(a - b)$ is an integer

$\Rightarrow (b - a)$ is an integer

$\Rightarrow (b, a) \in R$

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$

Therefore, R is symmetric.

Transitive Relation: A relation R defined on a set A is said to be transitive, if
$(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R, \forall x, y, z \in A$
or $xRy, yRz \Rightarrow xRz, \forall x, y, z \in R$.

Example :

Let A = { 1, 2, 3 } and R be a relation defined on set A as "is less than" and R = {(1, 2), (2, 3), (1, 3)} Verify R is transitive.

Solution :

From the given set A, let

a = 1, b = 2, c = 3

Then, we have

(a, b) = (1, 2) -----> 1 is less than 2

(b, c) = (2, 3) -----> 2 is less than 3

(a, c) = (1, 3) -----> 1 is less than 3

That is, if 1 is less than 2 and 2 is less than 3, then 1 is less than 3.

More clearly,

1R2, 2R3 -----> 1R3

Clearly, the above points prove that R is transitive.

Important Note :

For a particular ordered pair in R, if we have (a, b) and we don't have (b, c), then we don't have to check transitive for that ordered pair.

So, we have to check transitive, only if we find both (a, b) and (b, c) in R.

Equivalence relation: A relation R in a set A is said to be an equivalence relation if R is reflexive, symmetric and transitive.

Example: Show that the relation R is an equivalence relation in the set A = { 1, 2, 3, 4, 5 } given by the relation R = { (a, b):|a-b| is even }.

Solution :

R = { (a, b):|a-b| is even }. Where a, b belongs to A

Reflexive Property :

From the given relation,

|a – a| = | 0 |=0

And 0 is always even.

Thus, |a-a| is even

Therefore, (a, a) belongs to R

Hence R is Reflexive

Symmetric Property :

From the given relation,

|a – b| = |b – a|

We know that |a – b| = |-(b – a)|= |b – a|

Hence |a – b| is even,

Then |b – a| is also even.

Therefore, if (a, b) ∈ R, then (b, a) belongs to R

Hence R is symmetric

Transitive Property :

If |a-b| is even, then (a-b) is even.

Similarly, if |b-c| is even, then (b-c) is also even.

Sum of even number is also even

So, we can write it as a-b+ b-c is even

Then, a – c is also even.

So,

|a – b| and |b – c| is even , then |a-c| is even.

Therefore, if (a, b) ∈ R and (b, c) ∈ R, then (a, c) also belongs to R

Hence R is transitive.

Hence, the relation is Equivalence relation.

Function:

Let X and Y be two non-empty sets. A function or mapping f from X into Y written as f : X → Y is a rule by which each element x ∈ X is associated to a unique element y ∈ Y. Then, f is said to be a function from X to Y.

The elements of X are called the domain of f and the elements of Y are called the co-domain off. The image of the element of X is called the range of X which is a subset of Y.

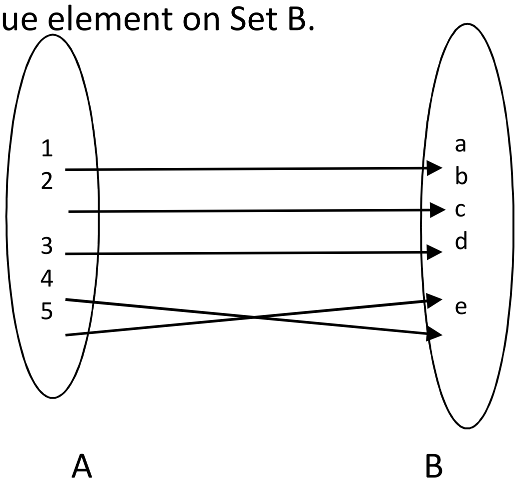
Note: Every function is a relation but every relation is not a function.

Types of Functions:

One-one Function or Injective Function: A function f: X → Y is said to be a one-one function, if the images of distinct elements of x under f are distinct, i.e. $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$, $\forall\, x_1, x_2 \in X$

OR

If we have to sets, set A and set B , then Injective can be defined as each element of Set A has a unique element on Set B.



A function which is not one- one, is known as many-one function.

Examples of Injective Function:

- The identity function X → X, is always injective.
- If function f: R→ R, then f(x) = 2x is injective.
- If function f: R→ R, then f(x) = 2x+1 is injective.

Important Note :

An injective function can be determined by the horizontal line test or geometric test.

1. If a horizontal line can intersect the graph of the function, more than one time then the function is not mapped as one-to-one.
2. If a horizontal line can intersect the graph of the function only a single time, then the function is mapped as one-to-one.

Example: Is Parabola a one to one function?

Solution : No, a parabola is not a 1-1 function. It can be proved by the horizontal line test.

A parabola is represented by the function $f(x) = x^2$

Now, if we draw the horizontal lines, then it will intersect the parabola at two points in the graph. Hence, for each value of x, there will be two output for a single input.

Example: Show that f: R→ R defined as $f(a) = 3a^3 – 4$ is one to one function?

Solution : Let $f (a_1) = f (a_2)$ for all $a_1 , a_2 \in R$

so $3a_1^3 – 4 = 3a_2^3 – 4$

$a_1^3 = a_2^3$

$a_1^3 – a_2^3 = 0$

$(a_1 – a_2) (a_1 + a_1a_2 + a_2^2) = 0$

$a_1 = a_2$ and $(a_1^2 + a_1a_2 + a_2^2) = 0$

$(a_1^2 + a_1a_2 + a_2^2) = 0$ is not considered because there is no real values of a_1 and a_2.

Therefore, the given function f is one-one.

Onto Function or Surjective Function: A function f : X → Y is said to be onto function or a surjective function, if every element of Y is image of some element of set X under f, i.e. for every y ∈ y, there exists an element X in x such that f(x) = y.

In other words, a function is called an onto function, if its range is equal to the co-domain.

Properties of a Surjective Function (Onto)

- We can define onto function as if any function states surjection by limit its co-domain to its range.
- The domain is basically what can go into the function, co-domain states possible outcomes and range denotes the actual outcome of the function.
- Every onto function has a right inverse
- Every function with a right inverse is a surjective function
- If we compose onto functions, it will result in onto function only.

Example: Let A={1,5,8,9) and B{2,4} And f={(1,2),(5,4),(8,2),(9,4)}. Then prove f is a onto function.

Solution: From the question itself we get,

A={1,5,8,9)

B={2,4}

& f={(1,2),(5,4),(8,2),(9,4)}

So, all the element on B has a domain element on A or we can say element 1 and 8 & 5 and 9 has same range 2 & 4 respectively.

Therefore, f: A → B is an surjective fucntion.

Bijective or One-one and Onto Function: A function $f : X \rightarrow Y$ is said to be a bijective function if it is both one-one and onto.

How to Prove that the Functions are Bijective?

Let's take a example to understand it –

Example: Show that the function f(x) = 3x – 5 is a bijective function from R to R.

Solution: Given Function: f(x) = 3x – 5

To prove: The function is bijective.

According to the definition of the bijection, the given function should be both injective and surjective.

(i) To Prove: The function is injective

In order to prove that, we must prove that f(a) =c and f(b)=c then a=b.

Let us take,

f(a) =c and f(b)=c

Therefore, it can be written as:

c = 3a-5 and c = 3b-5

Thus, it can be written as:

3a-5 = 3b -5

Simplify the equation; we will get

a = b

Thus, the given function is injective

(ii) To Prove: The function is surjective

To prove this case, first, we should prove that that for any point "a" in the range there exists a point "b" in the domain s, such that f(b) =a

Let, a = 3x -5

Therefore, b must be (a+5)/3

Since this is a real number, and it is in the domain, the function is surjective.

Thus, the given function satisfies the condition of one-to-one function, and onto function, the given function is bijective.

Hence, proved.

Composition of Functions:

Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Then, composition of functions f and g is a function from X to Z and is denoted by fog and given by $(fog)(x) = f[g(x)]$, $\forall\ x \in X$.
Note
(i) In general, $fog(x) \neq gof(x)$.
(ii) In general, gof is one-one implies that f is one-one and gof is onto implies that g is onto.
(iii) If $f : X \to Y$, $g : Y \to Z$ and $h : Z \to S$ are functions, then $ho(gof) = (hog)of$.

Example : Find gof and fog, if $f : R \to R$ and $g : R \to R$ are given by $f(x) = \cos x$ and $g(x) = 3x^2$. Show that $gof \neq fog$.

Solution: We have $gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$. Similarly, $fog(x) = f(g(x)) = f(3x^2) = \cos(3x^2)$. Note that $3\cos^2 x \neq \cos 3x^2$, for $x = 0$. Hence, $gof \neq fog$

Invertible Function:

A function $f : X \to Y$ is said to be invertible, if there exists a function $g : Y \to X$ such that $gof = I_x$ and $fog = I_y$. The function g is called inverse of function f and is denoted by f^{-1}.
Note:-
(i) To prove a function invertible, one should prove that, it is both one-one or onto, i.e. bijective.
(ii) If $f : X \to V$ and $g : Y \to Z$ are two invertible functions, then gof is also invertible with $(gof)^{-1} = f^{-1}og^{-1}$

Example : Let S = {1, 2, 3}. Determine whether the functions f : S → S defined as below have inverses. Find f –1, if it exists. (a) f = {(1, 1), (2, 2), (3, 3)} (b) f = {(1, 2), (2, 1), (3, 1)} (c) f = {(1, 3), (3, 2), (2, 1)}

Solution:

(a) It is easy to see that f is one-one and onto, so that f is invertible with the inverse f –1 of f given by f –1 = {(1, 1), (2, 2), (3, 3)} = f.

(b) Since f (2) = f (3) = 1, f is not one-one, so that f is not invertible.

(c) It is easy to see that f is one-one and onto, so that f is invertible with f –1 = {(3, 1), (2, 3), (1, 2)}.

Binary Operation:

A binary operation * on set X is a function * : X × X → X. It is denoted by a * b.

Commutative Binary Operation: A binary operation * on set X is said to be commutative, if a * b = b * a, ∀ a, b ∈ X.

Associative Binary Operation: A binary operation * on set X is said to be associative, if a * (b * c) = (a * b) * c, ∀ a, b, c ∈ X.
Note: For a binary operation, we can neglect the bracket in an associative property. But in the absence of associative property, we cannot neglect the bracket.

Identity Element: An element e ∈ X is said to be the identity element of a binary operation * on set X, if a * e = e * a = a, ∀ a ∈ X. Identity element is unique.
Note: Zero is an identity for the addition operation on R and one is an identity for the multiplication operation on R.

Invertible Element or Inverse: Let * : X × X → X be a binary operation and let e ∈ X be its identity element. An element a ∈ X is said to be invertible with respect to the operation *, if there exists an element b ∈ X such that a * b = b * a = e, ∀ b ∈ X. Element b is called inverse of element a and is denoted by a^{-1}.
Note: Inverse of an element, if it exists, is unique.

Example: For each binary operation * defined below, determine whether * is commutative or associative:

(i) On Z, define $a*b=a-b$

(ii) On Q, define $a*b=\frac{ab}{2}$

(iii) On Z , define $a*b=a^b$

(iv) On R – {– 1}, define $a*b=\frac{a}{b+1}$

Solution. (i) For commutativity: $a*b=a-b$ and $b*a=b-a$ = $-(a-b)\neq a*b$

For associativity: $a*(b*c)=a*(b-c)$ = $a-(b-c)=(a-b+c)$

Also, $(a*b)*c=(a-b)*c$ = $(a-b-c)$

$\therefore\ a*(b*c)\neq(a*b)*c$

Therefore, the operation * is neither commutative nor associative.

(ii) For commutativity: $a*b=\frac{ab}{2}$ and $b*a=\frac{ba}{2}=\frac{ab}{2}$ = $a*b$

For associativity: $a*(b*c)=a*\left(\frac{bc}{2}\right)$ = $\frac{abc/2}{2}=\frac{abc}{4}$

Also, $(a*b)*c=\left(\frac{ab}{2}\right)*c$ = $\frac{abc/2}{2}=\frac{abc}{4}$

$\therefore\ a*(b*c)=(a*b)*c$

Therefore, the operation * is commutative and associative.

(iii) For commutativity: $a*b=a^b$ and $b*a=b^a$

$\Rightarrow\ a*b\neq b*a$

For associativity: $a*(b*c)=a*b^c$ = $(a)^{b^c}$

Also, $(a*b)*c=(a^b)*c$ = a^{bc}

$\therefore\ a*(b*c) \neq (a*b)*c$

Therefore, the operation * is neither commutative nor associative.

(iv) For commutativity: $a*b=\dfrac{a}{b+1}$ and $b*a=\dfrac{b}{a+1} \Rightarrow a*b \neq b*a$

For associativity: $a*(b*c)=a*\left(\dfrac{b}{c+1}\right)$ = $\dfrac{a}{\dfrac{a}{c+1}+1}=\dfrac{a(c+a)}{b+c+1}$

Also, $(a*b)*c=\left(\dfrac{a}{b+1}\right)*c$ = $\dfrac{\cancel{a}/b+1}{c+1/\cancel{c}}=\dfrac{a}{(b+1)(c+1)}$

$\therefore\ a*(b*c) \neq (a*b)*c$

Therefore, the operation * is neither commutative nor associative.

Questions and Answers

1 mark each:

1. Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive.

Solution:
R = {(a, b) : b = a + 1}
R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}
When b = a, a = a + 1: which is false, So R is not reflexive.
If (a, b) = (b,a), then b = a+1 and a =b+1: Which is false, so R is not symmetric.
Now, if (a, b), (b,c) and (a, c) belongs to R then
b = a+1 and c =b+1 which implies c = a + 2: Which is false, so R is not transitive.
Therefore, R is neither reflexive, nor symmetric and nor transitive.

2. Give an example of a relation. Which is Symmetric but neither reflexive nor transitive.

Solution:
(i)Consider a relation R = {(1, 2), (2, 1)} in the set {1, 2, 3}
(x, x) $\notin$ R. R is not reflexive.
(1, 2) $\in$ R and (2,1) $\in$ R. R is symmetric.
Again, (x, y) $\in$ R and (y, z) $\in$ R then (x, z) does not imply to R. R is not transitive.
Therefore, R is symmetric but neither reflexive nor transitive.

3. Find gof and fog, if f(x) = | x | and g(x) = | 5x – 2 |

Solution:

f(x) = | x | and g(x) = | 5x – 2 |
gof =(gof)(x) = g(f(x) = g(|x|) = |5 |x| - 2|
fog = (fog)(x) = f(g(x)) = f(| 5x – 2 |) = ||5x – 2|| = | 5x – 2 |