Toppersnotes
Unleash the topper in you

CSIR-NET

Council of Scientific & Industrial Research

MATHEMATICAL SCIENCE

VOLUME - III

SIERRA
INNOVATIONS

INDEX

Caylay's Theorem :—

Every finite group of order n is isomorphic to some subgp of S_n

$f: S_n \longrightarrow A_{n+2}$.

$$f(\sigma) = \begin{cases} \sigma & : \sigma \text{ is even.} \\ \sigma\,(n+1\; n+2) & \sigma \text{ is odd.} \end{cases}$$

clearly f is 1-1 homo.

$S_n \cong f(S_n) < A_{n+2}$.

" S_n is isomorphic to some subgp of A_{n+2}" / A_{n+3}, A_{n+4}, - - - - -

There every group of order n is isomorphic to some subgp of A_{n+2} / A_{n+3}, A_{n+4}, - - - - -

Generalized Caylay's Theorem (G.C.T)

If G is finite group and G has proper subgroup of index n. (say H) then $\exists\ \phi : G \xrightarrow{\text{homo}} S_n$

s.that

$\ker\phi \subseteq H$

** corollary

Let p be the smallest prime divisor of order of G and if there exist a subgp of index p (say H) then H is normal in G.

Index Theorem :.

If G is a finite group and G has proper subgp of index n and $O(G) \nmid n!$.

$\Rightarrow$ G can not be simple.

If G is finite, ...

and $ind_G H = n$.

<u>By G.C.T.</u>

$\phi : G \longrightarrow S_n$.

s.that $\ker\phi \subseteq H$.

if $\ker\phi = \{e\}$.

By F.T.H.

$G \cong \phi(G) < S_n$.

$\Rightarrow o(G) \mid n!$

If $o(G) \nmid n!$

$\Rightarrow \ker\phi \neq \{e\}$.

And $\ker\phi \neq G$ $\quad (\because \ker\phi \subseteq H)$

$\ker\phi$ is proper normal subgp of G.

$\Rightarrow$ G can not be simple gp

<u>Example:-</u>

$o(G) = 108 = 2^2 \cdot 3^3$

$H < G$ and $o(H) = 3^3$

$ind_G H = 4$.

$o(G) \nmid 4!$

$\Rightarrow 108 \nmid 4!$

$\Rightarrow$ G can not be simple gp.

Example:- $o(G) = 24 = 2^3 \cdot 3$.

$H < G$ and $o(H) = 2^3$

$ind_G H = 3$.

$24 \nmid 3!$

$\Rightarrow$ G can not be simple.

Emmebeding Theorem: If G is finite simple group and G has a proper subgp of index n then G is isomorphic to some subgp of A_n.

If G is finite simple gp and $ind_G H = n$.

By G.C.T.

$\exists\ \phi : G \xrightarrow[homo]{} S_n$

$ker\phi \subseteq H$.

$ker\phi = \{e\}$. $\quad ker\phi = G$ $\times$

$\therefore ker\phi = \{e\}$.

By F.T.H.

$G \cong \phi(G) < S_n$.

supose $\phi(G)$ contains exactly half even and half odd.

$K = \{\sigma : \sigma$ is even $\}$

clearly $K \lhd \phi(G)$

$ind_G H = \frac{o(\phi(G))}{o(K)} = 2$.

$\Rightarrow \quad K \trianglelefteq \phi(G)$

$\phi(G)$ can't be simple

But $G \cong \phi(G) \quad (\because G$ is simple$)$

$\cancel{\times}$

$\therefore \ \phi(G)$ contains all even permutation

$G \cong \phi(G)$ and $o(G) \mid n!/2$

<u>corollarly.</u> If G is finite group and has a <u>proper</u> subgp of index n and $o(G) \nmid n!/2$

$\Rightarrow G$ can not be simple.

<u>corollary :</u>

If G is a finite group and has a proper subgp of index < 5. then G can not be simple

<u>case-I</u>

$ind_G H = 2$

$H \trianglelefteq G$.

$\Rightarrow \ G$ can not be simple.

<u>case-II</u>

$ind_G H = 3$.

Suppose G is simple.

$G \cong K < A_3$.

$ind_G H = 3$.

$o(G) = 3 \cdot o(K)$.

$o(G) \geq 6$

$o(K) \geq 6$

$K < A_3$. $\quad \cancel{\times}$

G can not be simple.

$ind_G H = 4$

suppose G is simple.

$G \cong K < A_4$.

$o(G) \geq 8$.

$o(K) \geq 8$.

and $o(K) \mid 12$ $\Rightarrow$ $o(K) = 12$.

$\therefore A_4 = K$

$G \cong A_4$ ✗. ($\because A_4$ is not simple)

G is cannot be simple.

Example:-

$G = A_5$.

$o(G) = 60$

A_5 = simple.

Possible order of subgp $\rightarrow$ 1, 2, 3, 4, 5, 6, 10, 12

15, 20, 30, 60
✗ ✗ ✗

Through out possible order of proper subgp $\rightarrow$ 12.

Note: If G is simple group of order 60 then G is isomorphic to A_5.

Sylow theorem

p-group:- A group G is said to be p-group if $\forall a \in G\setminus\{e\}, \exists\ n \in \mathbb{N}$ s.t. $o(a) = p^n$.

Example:- Q_8 is 2-group.

(ii) $(P(\mathbb{N}), \Delta)$ is 2-group.

(iii) $\mathbb{Z}_7 \times \mathbb{Z}_7$ is 7-group.

(iv) $H = \left\{ \frac{m}{2^n} + \mathbb{Z} : m \in \mathbb{Z},\ n \in \mathbb{N}\cup\{0\} \right\}$

2-group.

v $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{Z}_p \right\}$

$o(G) = p^3$, G is non-abelian group.

$\therefore$ G is p-group.

$o(Z(G)) = p$.

Note :- A finite group is said to be p-group iff $o(G) = p^n$ for some $n \in \mathbb{N}$.

p-Sylow's subgp (p-SSG).

Let H be subgroup of G s.t. $o(H) = p^m$ and $p^{m+1} \nmid o(G)$ then the subgroup of order p^m is defined as p-SSG.

Example:- $o(G) = 108 = 2^2 \cdot 3^3$

order of 2-SSG $= 2^2$.

order of 3-SSG $= 3^3$.

Theorem:-

order of $GL(n, Z_p) = (p^n - p^{n-1})(p^n - p^{n-2}) \cdots (p^n - 1)$

$= p^{n-1} p^{n-2} \cdots p^2 p \, (p-1)(\cdots)(p^n - 1)$

$= p^{\sum n-1} \cdot k.$

$$\boxed{\text{order of } p\text{-SS } G = p^{\sum n-1}}$$

$$\boxed{\therefore \text{ order of } p\text{-SS}G \text{ in } GL(n, Z_p) = \text{order of } p\text{-SS}G \text{ in } SL(n, Z_p) = p^{\sum n-1}}$$

Example:- No. of 5-SS G of S_6 is

(i) 16 (ii) 6 (iii) 36 (iv) $\cancel{1}$

$= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 2^4 \cdot 3^2 \cdot 5$

$= 720$

120

$5 \mid 720$ $5^{1+1} \nmid 720$

order of 5-SSG in $S_6 = 5$.

No. of 5-SSG in $S_6 = \dfrac{6 \times \cancel{4} \times 3 \times 2}{\cancel{4}}$

$= 36$

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Normalizer of a Subgroup:-

Let $H \leq G$.

Then normalizer of H is defined as

$$N(H) = \{x \in G : xHx^{-1} = H\}.$$

clearly $N(H) \leq G$.

Properties

(i) $H \trianglelefteq N(H)$

(ii) $H \trianglelefteq G \iff N(H) = G$.

(iii) If $H \trianglelefteq K$ then $K \subseteq N(H)$

(iv) $\dfrac{o(G)}{o(N(H))} = |cl(H)| = ind_G N(H)$, G is finite.

Sylow's First Theorem:

If $p^{\alpha} \mid o(G)$ then $\exists$ at least one subgp of order p^{α}.

Example: $o(G) = 108 = 2^2 \cdot 3^3$.

There is subgroup of order $= 2$

$= 4$

$= 3$

$= 3^2$

$= 3^3$.

Application:-

$o(G) = p^n$, $n \neq 1$

By Cyclic 1st Theorem.

$\exists H < G$ s.t $o(H) = p^{n-1}$

$ind_G H = p$.

$\Rightarrow H \trianglelefteq G$ By (G.C.T)

$\Rightarrow G$ can't be simple.

Problem :- Let G be a group of order 125, which of the following statements are necessarily true.

(i) G has a non-trivial abelian subgroup ✓

(ii) Z(G) is proper subgp. ✗

(iii) Z(G) has order 5. ✗

(iv) There is a subgp of order 25 ✓

$o(G) = 125$

abelian — non-abelian

$G = Z(G)$ — $o(Z(G)) = 5$.

$= 5^3$

By Ist Sylow's Theorem

There is a subgp $= 5$

$= 25$.

Sylow second Theorem:

All p - SSG are conjugate.

* **corollary :-**

p - SSG is unique iff p-SSG is normal.

Sylow III Theorem :

$$\text{No. of } p\text{-SSG} = 1 + kp \;\Big|\; \frac{o(G)}{p^m}$$

where p^m is order of p-SSG

$k = 0, 1, 2, 3, \ldots$

Example:- $O(G) = 56$

order of 2-SS G $= 2^3$

order of 7-SS G $= 7$.

No. of 2-SS G $= 1+2k \mid 7$

$= 1, 7$

No. of 7-SS G $= 1+7k \mid 8$

$= 1, 8$.

case-I

No. of 7-SS G $= 1$.

$\Rightarrow$ 7-SS G is unique.

$\Rightarrow$ 7-SS G is normal.

$\Rightarrow$ G cannot be simple.

case-II

No. of 7-SSG $= 8$

$H_1, H_2, \ldots, H_8$.

$O(H_i) = 7$.

total no. of elements of order $7 = 6 \times 8 = 48$.

$\Rightarrow$ 2-SS G is unique.

$\Rightarrow$ 2-SS G is normal

$\Rightarrow$ G can not be simple

Hence $O(G) = 56$ can not be simple group

Example:- $o(G) = 2 \times 3 \times 5$.

order 2-SSG = 2

order 3-SSG = 3

order 5-SSG = 5

No. of 2-SSG = $1+2k \mid 15$

$= 1, 3, 5, 15$.

No. of 3-SSG $= 1+3k \mid 10$

$= 1, 10$

No. of 5-SSG $= 1+5k \mid 6$

$= 1, 6$

suppose No. of 3-SSG = 10

No. of 5-SSG = 6

$H_1, H_2, \ldots H_{10}$ are 3-SSG.

$o(H_1) = 3$

total no. of elemets of order $3 = 2 \times 10 = 20$

$K_1, K_2 \ldots K_6$ are 5-SSG.

$o(K_1) = 5$.

total no. of elemets order $5 = 4 \times 6 = 24$

total no. of elemets of order 3 and 5

$= 20 + 24 = 44$

Either 2- SSG ...

$\Rightarrow$ G can't be simple.

Theorem:- $O(G) = p \cdot q$, $p<q$, and $p \nmid q-1$ then G is cyclic.

order of p-SSG = p

order of q-SSG = q.

No. of q-SSG $= 1+qk \mid p$

$= 1$

No. of p-SSG $= 1+pk \mid q$

$= 1, q$.

suppose No. of p-SSG = q.

$1+pk = q$

$p \cdot k = q-1$

$\Rightarrow p \mid q-1$ ✗

$\therefore$ No. of p-SSG = 1

Let H be p-SSG and K be q-SSG.

$H \trianglelefteq G$, $K \trianglelefteq G$.

$o(H) = p$, $o(K) = q$.

$H \cap K = \{e\}$.

$\exists$ $a \in G$ and $b \in G$ s.t.

$H = \langle a \rangle$ and $K = \langle b \rangle$.

In particular, $ab = ba$.

$$\langle a \rangle \cap \langle b \rangle = \{e\}.$$

$$o(ab) = l.c.m(o(a), o(b))$$

$$= pq. = o(G)$$

$\Rightarrow$ G is cyclic group.

Application:-

$$o\left(\frac{G}{Z(G)}\right) \neq pq, \quad p<q \quad \text{and} \quad p \nmid q-1.$$

Example $o(G) = 60 = 2^2 \times 3 \times 5.$

$$o(Z(G)) = 4.$$

$$o\left(\frac{G}{Z(G)}\right) = 15 \quad \times$$

$o(Z(G))$ can not be four.

Theorem:- $o(G) = p \cdot q, \quad p<q \text{ and } p \mid q-1$

G $\cong Z_{pq}$

G — Non-abelian group of order pq.

Problem:- The total of no of non-isomorphic gp of order 122 is

(a) 2 (b) 1 (c) 61 (d) = 4

$o(G) = 122 = 2 \times 61$

Z_{122}.

non-abelian of order 122

Problem :- Let G be non abelian group of order can.

(a) 25 ✗ (b) 55 (c) 125 (d) 35 ✗.

Example:- Every group of order 30 has cyclic subgp of order 15.

$o(G) = 30 = 2 \times 3 \times 5$.

No. of 3-SS G = 1, 10

No. of 5-SS G = 1, 6

Case-I

No. of 3-SS G = 1.

No. of 5-SS G = 6

Let H be 3-SSS and $o(H) = 3$
$H \Delta G$.

Let K be 5-SS G and $o(K) = 5$.

we know that $H \Delta G$, $K \leq G$.
$\Rightarrow HK < G$.

$o(HK) = 15$.

Case-II

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upto isomorphis: group of order 30.

Z_{30} — abelian

D_{15} → Non-abelian

$S_3 \times Z_5$ → Non-abelian

$D_5 \times Z_3$ → Non-abelion.

S_5 x S_7! S_6

No subgroup order is, cyclic.

$\frac{720}{30 \times 2} = 12$

$\boxed{\frac{7!}{30 \times 2}}$

Theorem:- Every group of order pqr can't be simple.

$O(G) = p \cdot q \cdot r$, $p < q < r$.

order of p-SSG $= p$

order q- SSG $= q$

order r- SSG $= r$

No. of p-SSG $= (1+pk) | qr$

$= 1, r, q, qr$

No. of q-SSG $= (1+qk) | pr$

$= 1, r, pr$

No. of r-SSG $= 1+rk | pq$

$= 1, pq$.

Suppose no. of p-SSG $= q$

no. of qSS $= r$

no. of r-SSG $= pq$.