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CSIR-NET

Council of Scientific & Industrial Research

PHYSICAL SCIENCE

VOLUME - II

CLASSICAL MECHANICS

SIERRA INNOVATIONS

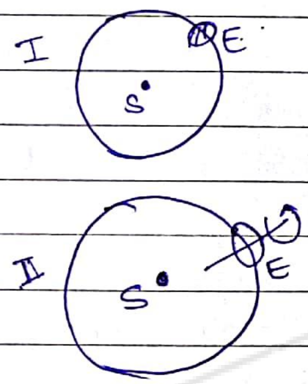
CLASSICAL MACHINES

Basic Definition –

diff. b/w particle and body –

The consideration of body and a particle depends on motion of a system.

e.g. in earth sun system - if earth revolving about the sun without spinning it is treated as a particle but when spinning of earth on its own axis will be part of dynamics then the shape of earth matters.



State of a classical system :-

At particular time t if one can find position and its conjugate momentum

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$p_x = m\dot{x}$$

eqⁿ of motion :-

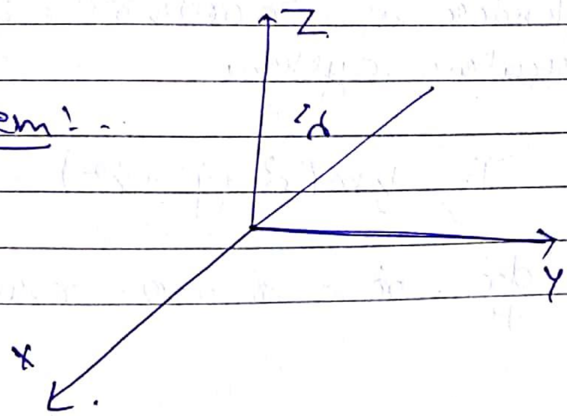
The relationship b/w position, velocity and acc. at particular time is eqⁿ of motion.

Co-Ordinate System :-

Cartesian Co-Ordinate System :-

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v} = \dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$



$\vec{\dot{x}}$ — Conjugate velocity with $\hat{x}$ dir^n.

$$K.E. = \frac{1}{2} m |v|^2$$

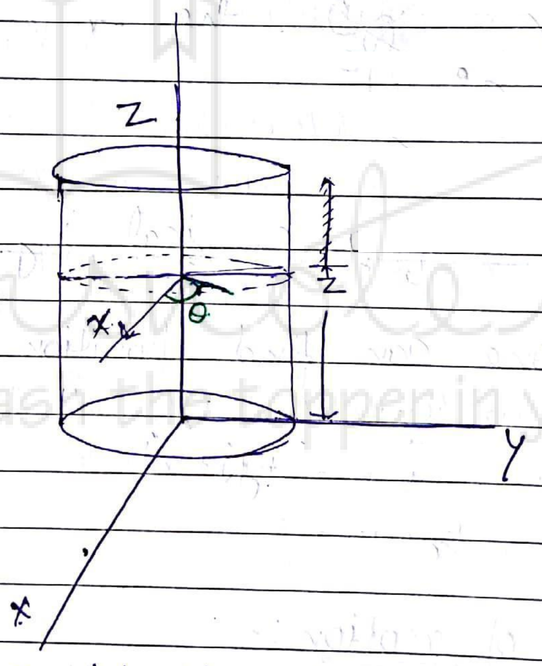
$$= \frac{1}{2} m (\vec{v} \cdot \vec{v})$$

$$= \frac{1}{2} m \, \vec{\dot{r}} \cdot \vec{\dot{r}}$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Cylendrical Co-ordinate :-

In cylendrical co-ordinate r is distance from z. and the angle θ is angle b/w r and x-axis



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

If $z = 0$ then $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$ identify circular polar system.

Cylender is coordinate to two identify symmtry of circular system.

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\frac{dx}{dt} = \dot{x} = \dot{r}\cos\theta - r\sin\theta \, \dot{\theta}$$

$\frac{dy}{dt} = \dot{y} = \cancel{\dot{r}\cos\theta +} \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

$\frac{dz}{dt} = \dot{z} = \dot{z}$

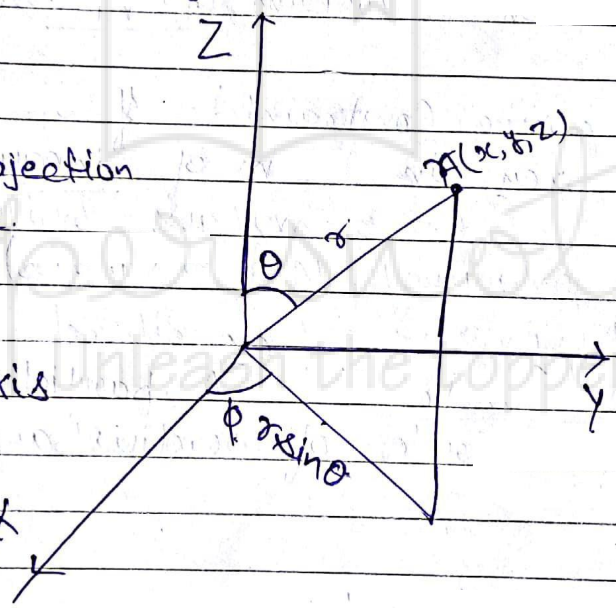
$$\boxed{T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right)}$$

($r\dot{\theta}$ is linear velocity and $\dot{\theta}$ is angular velocity)

~~Sperical Co-ordinate~~

Spherical Co-ordinate System :- (r, θ, ϕ)

$Z = r\cos\theta$



and $r\sin\theta$ is projection of r in x-y plane

$r\sin\theta$ is making angle ϕ with x-axis

$$\left.\begin{aligned} X &= r\sin\theta\cos\phi \\ Y &= r\sin\theta\sin\phi \\ Z &= r\cos\theta \end{aligned}\right\}$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \underbrace{r^2\sin^2\theta\,\dot{\phi}^2}_{\text{linear velocity}}\right)$$

$\dot{x} = \dot{r}\sin\theta\cos\phi + r\cos\theta\cos\phi\,\dot{\theta} - r\sin\theta\sin\phi\,\dot{\phi}$

$\dot{y} = \dot{r}\sin\theta\sin\phi + r\cos\theta\sin\phi\,\dot{\theta} + r\sin\theta\cos\phi\,\dot{\phi}$

$\dot{z} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta}$

Constraint motion :- limit to motion is constraint motion.

Constranit Eqⁿ - The relationship b/w variables is identified as eqⁿ of constraint.
There are two type of constraint eqⁿ -
1) Holonomic Constraint
2) Non-Holonomic Constraint.

1) **Holonomic Constraint** :- If relationship b/w variable can be expren in term of eqⁿ. $f(q_1, q_2, q_3) = 0$

2) **Non-Holonomic Constraint** :- If relationship b/w variables can be expren in term of inequlity.

$$f(q_1, q_2, q_3) \geq 0, \leq 0$$

Holonomic Constraint :- ex A particle is constraint to move in a circle of radius a in {x-y plane.

Degree Of Freedom :-
No. of independent motion is identified as degree of freedom.

If all particle are free -
x_1, y_1, z_1
x_2, y_2, z_2
⋮

$\boxed{\text{then d.o.f} = 3N}$

If k no. of constraints Holonomic.

$\boxed{\text{d.o.f.} = 3N - k}$

Qu Two particles constraint to move in x-y plane what will be d.o.f.

Solⁿ
x_1, y_1, z_1 — cont.
x_2, y_2, z_2

So. d.o.f. $= 3N-2$
$= 3\times2-2$
$= \underline{4}.$

Q two particles constraint to move in x-y plane and distance b/w them is fixed.

Solⁿ

$d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
↳ fix

and $z_1 = 0$, $z_2 = 0$
So $K = 3$
So. d.o.f. $3\times2-3$
$= \underline{3}$

Qu Two particles constraint to move in x-y plane and their distance b/w them is fixed and both particle are moving on a same curve. What will be d.o.f.

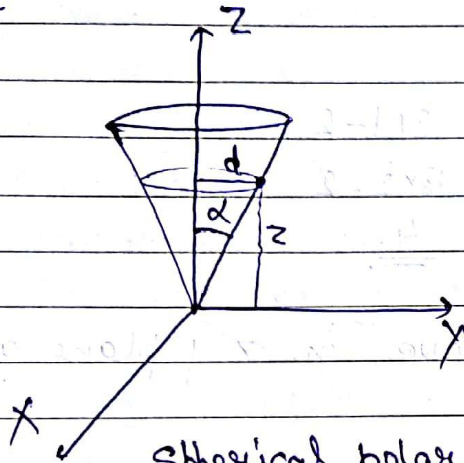
$y_1 = f(x_1)$
$y_2 = f(x_2)$] d-fix
$z_1 = 0$, $z_2 = 0$

So. d.o.f. $= \underline{1}$ $K = \underline{5}$

Qu A particle of mass m constraint to move in a inner surface of cone of half angle α. Write down eqⁿ of constraint in i) Cartisian co-ordinate system.
ii) Cylendrical system.
(iii) Spherical polar system.

and what will be degree of freedom.

Sol^n



$$\tan\alpha = \frac{d}{z}$$

$$= \frac{\sqrt{x^2+y^2}}{z}$$

$$\tan\alpha = \frac{r}{z}$$ (cylendrical co.ordinate)

Spherical polar co-ordinate system.

$$\tan\alpha = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

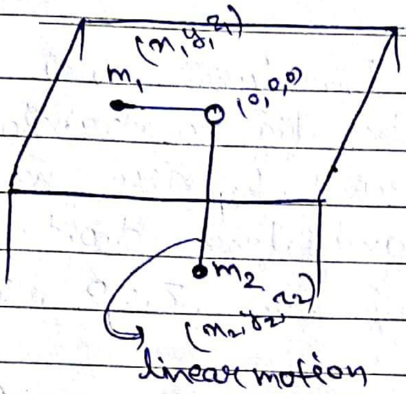
$$\underline{\alpha = \theta}$$

d.o.f. $= 3\times1-1$
$= 2$

eq^n of constraint - $z = \sqrt{x^2+y^2}\tan\alpha$
$= r\tan\alpha$

$\underline{\theta = \alpha}$ so sphere

Q. Two particles of mass m_1 and m_2 attached to end of string of length l. Where 1st particle m_1 constraint to move on a surface of table while another mass m_2 hang vertically & constraint to move in verticle dir^n under gravity. Write down eq^n of constraint and find d.o.f. of a ~~co~~ system.



$z_1 = 0$ — (1)
(mass m_1 constraint to move on the table)

$x_2 = 0$ — (2), $y_2 = 0$ — (3)
(mass m_2 constraint to move only on verticle dirn)

$\sqrt{x_1^2 + y_1^2} + z_2 = l \text{ (const.)}$ — (4)

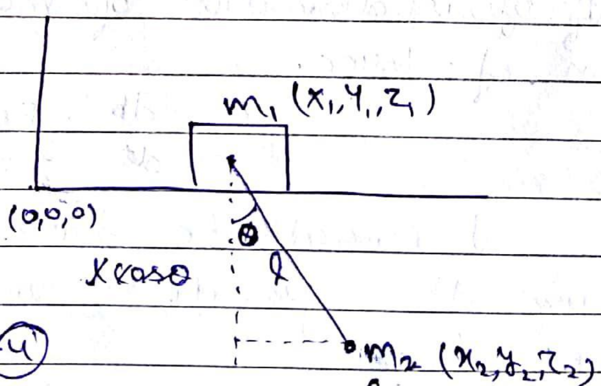
So 4 eqn of constraint so d.o.f. $= 3 \times 2 - 4$
$= 2$

$x_1^2 + y_1^2 = (l - z_2)^2$
circular polar motion

Moving Pendulum

Q. Two particle of mass m_1 & m_2 attached to massless rod where mass m_1 constraint to move in x-axis and mass m_2 can move in x-y plane under gravity in y-dirn as shown in fig. Write down eqn of constraint.

$y_1 = 0$ (1), $z_1 = 0$ (2)
$z_2 = 0$ (3)



$x_2 = x_1 + l\sin\theta$
$y_2 =$

$(x_2 - x_1)^2 + y_2^2 = l^2$ — (4)

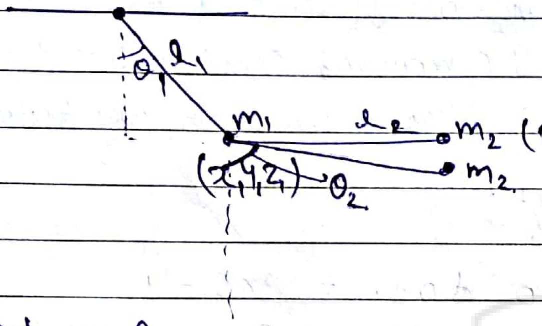
circular polar co-ordinate with center $(x_1, 0)$

d.o.f. $= 2$ $(3 \times 2 - 4)$

x

Double Pendulum -

double pendulum is shown in fig. constraint to move in x-y plane under gravity, l_1 and l_2 are manless rood write down eq^n of constraint and find d.o.f.



$z_1 = 0$, $z_2 = 0$

$x_2 = (x_1 + l)$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2$

d.o.f. = 2

$x_1^2 + y_1^2 = l_1$

Generlized Co-ordinate :- (q_i)

Any physical quantity which will define motion of the system uniquely is identified as generlized co-ordinate.

If q_i is generlized co-ordinate then $\dot{q}_i = \frac{dq_i}{dt}$ is identified as generlized velocity.

e.g. If q_i is dimension of momentum then $\dot{q}_i$ is dimension of force.

$$\dot{q}_i = \frac{dq}{dt} = \frac{dp}{dt} = F$$

If generlized co-ordinate have dimension of angular momentum then generlized velocity have dimension of Torque.

Hence generlized co-ordinate have min no. of variable which will analyse independent motion of a system. So no. of generlized co-ordinate must be equal to no. d.o.f.

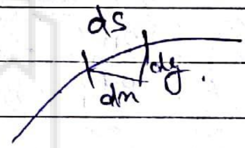
Mathematical Analogy of Classical (~~Vo~~ Calculus of variation)

$$\delta S = \underbrace{\delta\int f(y,\dot{y},x)\,dx = 0}_{\text{extreme Cond}^n} \qquad \dot{y} = \frac{dy}{dx}$$

If S is frictional extremum fu^n and $f(y,\dot{y},x)$ follow Euler lagrangian eq^n.

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0} \quad \text{Euler Langrangian eq}^n$$

$$dS = \sqrt{(dx)^2 + (dy)^2}$$



$$\delta S = \delta\int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$S = \delta\int\sqrt{1+\dot{y}^2}\,dx$$

$$f = \sqrt{1+\dot{y}^2}$$

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0$$

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = C \rightarrow \dot{y} = \text{const.}$$

$$y = mx + C$$

straight line motion

Principle of least action (Hamiltons Principle):-

The action is defined $\int L\,dt$
where L is fu^n of generalized coordinate, generalized velocity and time.

$$L = L(q,\dot{q},t)$$

Hamilton principle says a natural particle can follow those path which action is extremum.

$$\delta A = \delta \int L(q_i, \dot{q}_i, t)\, dt = 0$$

$f \to L$ $\qquad \delta S = \delta \int f(y, \dot{y}, x)\, dx = 0$

$y \to q_i$

$\dot{y} \to \dot{q}_i$

$x \to t$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \rightarrow \text{this is force eq}^n \text{ (vector)}$$

$$\boxed{L = T - V}$$

scalar quantity

Generlized Momentum :-

If lagrangian of a system $L(q_i, \dot{q}_i, t)$
The generlized Momentum p_i is defined as

$$\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$$

generlized momentum can be linear momentum and can be angular momentum depends on the dimens of generlized co-ordinate.

Cyclic Co-ordinate :- If lagrangian is not explicit fun of any perticular generlized co-ordinate let say q_j.

$$\boxed{\frac{\partial L}{\partial q_j} = 0} \rightarrow \text{then lagrangian is not explicit fun of } q_j$$

then q_j is identified as cyclic co-ordinate.

Conservation of momentum :- If q_j is cyclic co-ordinate then momentum corresponds to q_j is p_j will conserve during the motion.

e.g. $L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{1}{2}k(x^2+y^2)$

Q. Identify generlized co-ordinate
Q. Identify generlized velocity
Q. D.O.f.
Q. What will be the generlized Velocity
Q. Write down the cyclic co-ordinate.
Q. Discuss the constraint of motion.
Q. Write down eqn of motion.

Soln 1. x, y, z are generlized co-ordinate
2. $\dot{x}, \dot{y}, \dot{z}$ are generlized velocity.
3. D.O.f - 3
5. $\frac{\partial L}{\partial q} = 0$

$\frac{\partial L}{\partial z} = 0$ so z is a generlized ~~co~~ cyclic co-ordi

E.T. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$m\ddot{x} + kx = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow m\ddot{y} + ky = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow m\ddot{z} = 0$

Qn $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - \frac{1}{2}kr^2$

Cyclic Co-ordinate - $\frac{\partial L}{\partial \theta}$ $\frac{\partial L}{\partial \phi} = 0$

so ϕ is a cyclic co-ordinate.

eq^n of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$m\ddot{r} - (mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - kr) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$mr^2\ddot{\theta} - mr^2\sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

$$mr^2\sin^2\theta\,\ddot{\phi} = 0$$

If $\frac{\partial L}{\partial q_j} = 0$ and q_j is cyclic co-ordinate.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

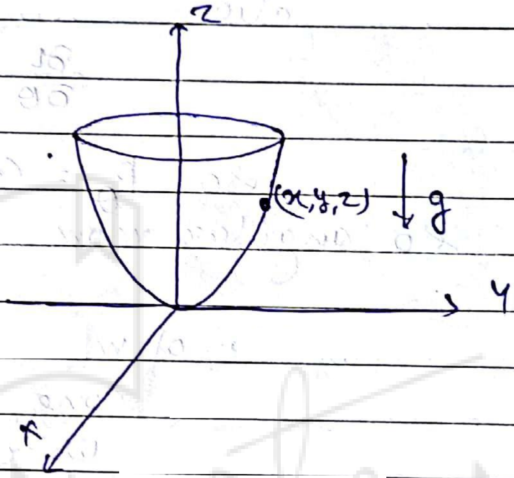
$$\frac{\partial L}{\partial \dot{q}_j} = \text{const.}$$

$$\boxed{p_j = \text{const.}}$$

No. of d.o.f must be equal to no. of generlized co-ordinate.

Q. A particle of mass m constraint to move in a inner surface of ~~ball~~ baul and eqn is given by $z = \frac{1}{2} b(x^2+y^2)$. The particle is moving under gravity in z-dirn. Write down lagrangian of system in suitable co-ordinate. Discuss Conservation law. Write down eqn of motion.

Soln



$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

$$V(n) = mgz$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$$

In Cylendrical System -

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) - mgz$$

$$z = \frac{1}{2}br^2$$

In Spherical polar system -

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) - mg\cdot\frac{1}{2}br^2\sin\theta$$

$$\left(z = \frac{1}{2}br^2\sin\theta\right)$$

In Cylendrical ~~co~~system system is more appropriate to solve instead of cartitian and spherical.

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + b^2r^2\dot{r}^2) - \frac{mg}{2}br^2$$

$$= \frac{1}{2}m\left[(1+b^2r^2)\dot{r}^2 + r^2\dot{\theta}^2\right] - \frac{mg}{2}br^2.$$

Eq^n of motion -

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\cancel{m((1+b^2r^2)\,\ddot{r}}$$

$$2mb^2 r\dot{r}\dot{r} + m(1+b^2r^2)\ddot{r} - mr\dot{\theta}^2 - mb^2 r\dot{r}^2 + 0 + mgbr = 0$$

$$m(1+b^2r^2)\ddot{r} + mb^2 r\dot{r}^2 - mr\dot{\theta}^2 + mgbr = 0$$

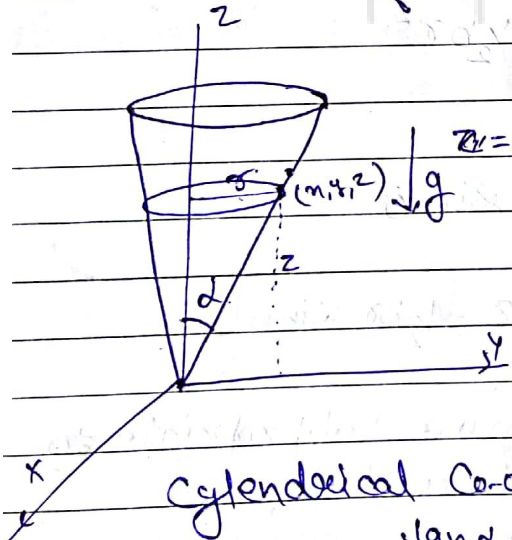
and θ is cyclic co-ordinate here.

$$\frac{\partial L}{\partial \theta} = 0$$

so p_θ = const

so angular momentum is conserve.

Qn A particle of mass m constraint to move in a innersurface of cone of half angle α. The motion is under gravity. Write down lagrangian of a system in suitable co-ordinate. Identify cyclic co-ordinate & conservation of momentum. Write down eq^n of motion



$\tan\alpha = \frac{r}{z} \Rightarrow z = r\cot\alpha$

$z = \sqrt{x^2+y^2}\,\cot\alpha$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - mgz$$

Cylendrical Co-ordinate system -

$\tan\alpha = r/z$

$L = \frac{1}{2}m(\dot{r}^2 + r^2$

Spherical polar -

$L = \frac{1}{2} m(\dot{r}^2 + \cancel{r^2\dot{\alpha}^2} + r^2 \sin^2\alpha\, \dot{\phi}^2) - mg\, r\, \cancel{\tan\alpha} \cos\alpha$

$L = \frac{1}{2} m(\dot{r}^2 + r^2 \sin^2\alpha\, \dot{\phi}^2) - mgr\cos\alpha.$

$\frac{\partial L}{\partial \phi} = 0$ So ϕ is a cyclic co-ordinate.

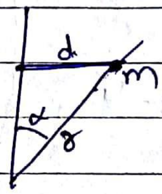
So $\frac{\partial L}{\partial \dot{\phi}} = \text{const}.$

$p_\phi = \text{const}.$

$mr^2 \sin^2\alpha\, \dot{\phi} = \text{const}.$

$\dot{\phi} = \frac{\text{Const}}{mr^2\sin^2\alpha}$

$= \frac{c}{I}$ - M.I.



$I = md^2$
$= mr^2 \sin^2\alpha$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$

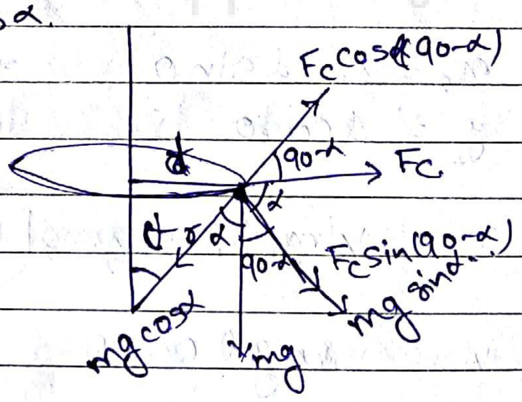
$\frac{d}{dt}(m\dot{r}) - mr\sin^2\alpha\, \dot{\phi}^2 + mg\cos\alpha = 0$

$m\ddot{r} = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha.$

$F = F_c \sin\alpha - mg\cos\alpha$

$F_c = m\omega^2 d$
$= m\dot{\phi}^2 r\sin\alpha.$

$F = mr\sin^2\alpha\,\dot{\phi}^2 - mg\cos\alpha$



At particular $r = r_0$
$m\ddot{r} = 0$

So there is not any external force in dir^n of r so particle will achive eq^m and revolve in circular motion.