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IES

GENERAL STUDIES

MATHS

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PROJECT MANAGEMENT

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QUALITY

VOLUME-I

Sierra Innovations Pvt. Ltd.

Contents

Maths

LINEAR ALGEBRA

Determinant Value of a Square matrix →

The sum of the products of elements of a row (column with their corresponding cofactors is known as determinant value of square matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 1 \\ 1 & 5 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 1 \\ 5 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$$

$= 1(1-5) - 3(1-1) + 2(5-1)$

$= 4$ Ans.

$$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}$$

$C_1 \longrightarrow C_1 + C_2 + C_3 + C_4$

$$\begin{vmatrix} 10+x & 2 & 3 & 4 \\ 10+x & 2+x & 3 & 4 \\ 10+x & 2 & 3+x & 4 \\ 10+x & 2 & 3 & 4+x \end{vmatrix}$$

$R_2 - R_1 \quad R_3 - R_1 \quad R_4 - R_1$

$$\begin{vmatrix} 10+x & 2 & 3 & 4 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}$$

$$= (10+x)\begin{vmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = (10+x)\,x^3$$

~~Trick~~

~~Q.~~

$$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)(x-a)(x-a)(x-a)$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4(1)(1) = 4$$

$\downarrow$

$2(3) - 1(2-1) + 1(1-2) = 4$

\# A Square matrix A is said to be

(1) Symmetric if $A^T = A$ i.e; $a_{ij} = a_{ji} \quad \forall\ i, j$

(2) skew symmetric if $A^T = -A$ i.e; $a_{ij} = -a_{ji} \quad \forall\ i, j$

(3) Orthogonal matrix if $AA^T = A^TA = I$

\# Every square matrix can be expressed as the sum of of a symmetric and skew symmetric matrix i.e; $A = \underbrace{\left(\frac{A+A^T}{2}\right)}_{\text{symmetric}} + \underbrace{\left(\frac{A-A^T}{2}\right)}_{\text{skew-symmetric}}$

$A = \begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$ $\qquad A^T = \begin{bmatrix} 2 & 6 \\ 3 & 7 \end{bmatrix}$

$\frac{A+A^T}{2} = \frac{1}{2}\begin{bmatrix} 4 & 9 \\ 9 & 14 \end{bmatrix} = \begin{bmatrix} 2 & 9/2 \\ 9/2 & 7 \end{bmatrix} \rightarrow$ Symmetric

$\frac{A-A^T}{2} = \frac{1}{2}\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 \\ 3/2 & 0 \end{bmatrix} \rightarrow$ skew symmetric.

Properties of Determinant →

$|A^T| = |A|$

$|AB| = |A||B|$

$|A+B| \neq |A| + |B|$

The determinant value of a triangular (or) a diagonal matrix is the product of its leading diagonal element

$A = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 4 & 6 \\ 0 & 0 & 8 \end{bmatrix}$ $\qquad |A| = 2(32-0) - 3(0-0) + 7(0-0)$
$= 64$

In a square matrix, if each element of a row (column) is zero then the value of its determinant is zero.

$A = \begin{bmatrix} 2 & 3 & 5 \\ 6 & 9 & 8 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad |A| = 2(0-0) - 3(0-0) + 5(0-0)$
$= 0$

In a square matrix, if two rows (columns) are identical / proportional then the value of its determinant is zero.

$$A = \begin{bmatrix} 2 & 3 & 7 \\ 6 & 9 & 8 \\ 6 & 9 & 8 \end{bmatrix}$$

$|A| = 2(72-72) - 3(48-48) + 7(54-54)$
$= 0$

The determinant value of a skew symmetric matrix of odd order is always zero.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}_{3\times3}$$

$|A| = 0 - 1(0+6) + 2(3-0)$
$= 0$

The determinant value of a non zero skew symmetric matrix of even order is always a perfect square.

$$A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

$|A| = 0 + 9 = 9$

Null matrix is the only matrix which is both symmetric as well as skew symmetric matrix.

The determinant value of an orthogonal matrix is always either 1 or −1

$AA^T = I$

$|AA^T| = |I|$

$|A||A^T| = |I|$

$|A||A| = 1$

$|A|^2 = 1$

$|A| = \pm 1$

If A is a square matrix of order n and k is any scalar then $|kA| = k^n|A|$

e.g.: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad kA = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}$

$$|kA| = \begin{vmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{vmatrix} = k^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = k^2|A|$$

If A is a non singular matrix of order n then

$|A| \neq 0$

(I) $A(\text{Adj } A) = |A| I$

(II) $A^{-1} = \dfrac{\text{Adj } A}{|A|}$

(III) $|\text{Adj } A| = |A|^{n-1}$

(IV) $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2}$

(V) $|A^{-1}| = \dfrac{1}{|A|}$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad$ cofactor matrix $= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$|A| = ad - bc$ $\quad$ $\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad$ $\text{Adj } A = [c_{ij}]^T$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$A = \begin{bmatrix} 5 & 3 \\ 7 & 8 \end{bmatrix}$ $\quad$ $|A| = 40 - 21 = 19$

$\text{Adj } A = \begin{bmatrix} 8 & -3 \\ -7 & 5 \end{bmatrix}$ $\quad$ $A^{-1} = \dfrac{\text{Adj } A}{|A|} = \dfrac{1}{19}\begin{bmatrix} 8 & -3 \\ -7 & 5 \end{bmatrix}$

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$|A| = -1(1-4) + 2(2+4) - 2(-4-2)$$
$$= (-1)(-3) + 2\times 6 - 2\times(-6)$$
$$= 3 + 12 + 12 = 27$$

Trick to find Adj A (only for 3x3)

$$\begin{array}{ccccc} M & 1 & -2 & -2 & 1 \\ L & -2 & 1 & -2 & -2 \\ F & 2 & 2 & -1 & 2 \\ M & 1 & -2 & -2 & 1 \end{array}$$

$$\text{Adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj } A}{|A|} = \frac{1}{27}\begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

Q. If $A_{m\times n}$ and $B_{n\times p}$ are multiplied then the total no. of multiplicative and additive operations are needed to get matrix AB

a) mpn, mpn
b) mpn, mp(n-1) ✓
c) mp(n-1), mpn
d) mpn, mpn-1

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \\ \\ \end{bmatrix}_{m\times n} \quad B = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}_{n\times}$$

$$AB = \begin{bmatrix} a_{11}\cdot b_{11} + a_{12}\cdot b_{21} + a_{13} * b_{31} + \cdots + a_{1n}\cdot b_{n1} \\ \\ \\ \end{bmatrix}_{m\times}$$

$\times$	$+$
n	$n-1$
mpn	$mp(n-1)$ Ans.

Q. If $A_{3\times4}$ and $B_{4\times5}$ are multiplied then the total no. of scalar operations are needed to get AB.

Sol^n $A_{3\times4}$ $B_{4\times5}$

multiplications $\longrightarrow$ $3\times5\times4 = 60$

Additions $\longrightarrow$ $3\times5\times3 = 45$

Total Scalar operations $= 105$

Q. If $A_{3\times4}$, $B_{4\times5}$ & $C_{5\times3}$ are multiplied then the minimum no. of multiplications are needed to get matrix ABC.

Sol^n $(AB)C$ | $A(BC)$

3×4 4×5 $\longrightarrow$ $3\times5\times4 = 60$ | 4×5 5×3 $\longrightarrow 4\times3\times5 = 6$

3×5 5×3 $\longrightarrow$ $3\times3\times5 = 45$ | 3×4 4×3 $\longrightarrow 3\times3\times4 = 3$

3×3 $\quad$ 105 | 3×3 $\quad$ 9

$\therefore$ minimum $= 96$.

Q. If $A = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}$ is expressed as $P+Q$ where P is symm. & Q is skew symm. then $|Q| =$ ______

Sol^n $Q = \dfrac{A - A^T}{2} = \dfrac{1}{2}\left\{\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}\right\}$

$= \dfrac{1}{2}\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -3/2 \\ 3/2 & 0 \end{bmatrix}$

$|Q| = \begin{vmatrix} 0 & -3/2 \\ 3/2 & 0 \end{vmatrix} = \dfrac{9}{4}$

Q. The no. of different $n \times n$ matrices with each element being 0 or 1.

a) 2^n b) 2^{n^2} c) $2^{\frac{n^2-n}{2}}$ d) $2^{\frac{n^2+n}{2}}$

Sol^n $\begin{bmatrix} & \\ & \end{bmatrix}_{2\times 2}$

a_{11}	a_{12}	a_{21}	a_{22}	
$\square$	$\square$	$\square$	$\square$	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
2	2	2	2	$= 2^4$

$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{n\times n}$ $\longrightarrow n$, $\longrightarrow n$, $\longrightarrow n$ $\qquad 2^{n^2}$

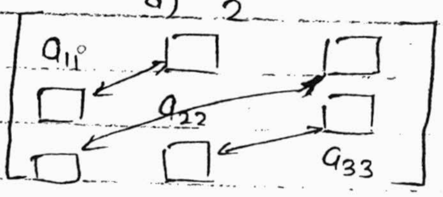
Q. The no. of different $n \times n$ symmetric matrices with each element being 0 or 1.

Sol^n a) 2^n b) 2^{n^2} c) $2^{\frac{n^2-n}{2}}$ d) $2^{\frac{n^2+n}{2}}$

2×2

$\begin{bmatrix} a & c \\ c & d \end{bmatrix}$

a_{11}		a_{12} & a_{21}		a_{22}	
$\square$		$\square$		$\square$	
2	$\times$	2	$\times$	2	$= 2^3$



a_{11} a_{22} a_{33}

$\bigcirc \; \bigcirc \; \bigcirc \; \square \; \square \; \square$

$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$

Q. If $A = (a_{ij})_{3\times3}$ Where $a_{ij} = i^2 - j^2$ $\forall$ i, j then |A| =

Sol^n $\begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix} \Rightarrow |A| = 0$

Q. Which of the following is correct for 3x3 matrix : P, Q, R

a) $P(Q+R) = PQ + RP$

b) $(P-Q)^2 = P^2 - 2PQ + Q^2$

✓ c) $(P+Q)^2 = P^2 + PQ + QP + Q^2$

d) $|P+Q| = |P| + |Q|$

Sol^n In matrix theory $AB \neq BA$

a) $P(Q+R) = PQ + PR$

b) $(P-Q)^2 = P^2 + Q^2 - PQ - QP$

d) $|P+Q| \neq |P| + |Q|$ Ans.

Q. If A, B, C, D, E are non singular matrices of same order such that DABEC = I then $B^{-1} =$

Sol^n DABEC

anticlockwise

$B^{-1} = ECDA$

Q let $M^4 = I$ ($M \neq I$, $M^2 \neq I$, $M^3 \neq I$) for any +ve integer k

Q. If X and Y are singular matrices such that XY = Y, YX = X then $X^2 + Y^2 =$ ____

Sol^n $X^2 + Y^2$

$XX + YY$

$\downarrow\downarrow \quad \downarrow\downarrow$

$X\,\underbrace{YX} + Y\,\underbrace{XY}$

$\downarrow\downarrow \quad \downarrow\downarrow$

$YX + XY$

$= X + Y$

Rank of a matrix → The order of highest ordered non-zero minor is called rank of the matrix.

Minor → The determinant value of square sub matrix.

e.g.;

$$A = \begin{bmatrix} 1 & -1 & 3 & 2 \\ 2 & 1 & 5 & 7 \\ 1 & 1 & 7 & 2 \end{bmatrix}_{3\times4}$$

$$3\times3 = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 1 & 5 \\ 1 & 1 & 7 \end{vmatrix} = 1(7-5) + 1(14-5) + 3(2-1)$$

$$= 14 \neq 0$$

∴ a 3x3 minor is non zero.

$\rho(A) = 3$. (Rank of matrix)

Q.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}_{3\times3}$$

3x3

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{vmatrix} = 1(14-12) - 1(7-3) + 1(4-2)$$

$$= 0$$

a 3×3 minor is zero.

$$2\times 2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2-1 = 1 \neq 0$$

$\therefore \rho(A) = 2$ Ans.

The rank of null matrix is taken as zero.

The rank of Non singular matrix is its order.

The rank of singular matrix is less than its order.

If A is $m\times n$ matrix then $\rho(A) \le \min\{m,n\}$

$A_{3\times 100}$ = $\rho(A) \le \min\{3, 100\}$

$\rho(A) \le 3$

$\rho(A) = \rho(A^T) = \rho(AA^T) = \rho(A^TA)$

$\rho(AB) \le \min\{\rho(A), \rho(B)\}$

If A and B are two matrices of same order then
$\rho(A+B) \le \rho(A) + \rho(B)$

$$A+B = \begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 3 & 7 \end{bmatrix}$$

$\rho(A+B) = 2$ $\qquad \rho(A) = 1$ $\qquad \rho(B) = 1$

$2 = 1 + 1$

$$A+B = \begin{bmatrix} 2 & 12 \\ 8 & 11 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 \\ 6 & 7 \end{bmatrix}$$

$2 < 2 + 2$

In a matrix if all rows (columns) are identical / proportional then its rank is always 1.

$A = (a_{ij})_{n\times n}$ where $a_{ij} = i\cdot j$ for all i, j

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 2 & \cdots & n \\ 2 & 4 & \cdots & 2n \\ 3 & 6 & \cdots & 3n \\ \vdots & \vdots & & \vdots \\ n & 2n & \cdots & n^2 \end{bmatrix}$$

All rows are proportional.

$\therefore \rho(A) = 1$

If A is $n\times n$ matrix with rank n then $\rho(\text{Adj}A) = n$

If A is $n\times n$ matrix with rank $n-1$ then $\rho(\text{Adj}A) = 1$

If A is $n\times n$ matrix with rank $n-2$ or less then $\rho(\text{Adj}A) = 0$

Q. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3\times 3}$

3×3 $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 1(-1) - 1(-2) + 1(-1)$

$= 0$

a 3×3 minor is zero

$2\times 2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$

2×2 minor is non zero $\therefore \rho(A) = 2$

M	2	3	1	2
L	3	4	1	3
F	1	2	1	1
M	2	3	1	2

$$\text{Adj}\,A = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

All rows are proportional

$\therefore \rho(\text{Adj}\,A) = 1$ Ans

Echelon Form →

The no. of zero's before non zero element in a row are less than such number of zero's in the next row.

Zero rows (if any) must follow non zero rows.

The number of non-zero rows is called rank of the matrix when it is in echelon form.

e.g.;

$$A = \begin{bmatrix} 1 & -1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

$$B = \begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(B) = 2$

$$C = \begin{bmatrix} 1 & -1 & 2 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$\rho(C) = 3$

$$D = \begin{bmatrix} 1 & -1 & 6 \\ 0 & 3 & 7 \\ 0 & 0 & 6 \end{bmatrix}$$

$\rho(D) = 3$

Rank of a matrix is unique

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 3 \\ 2 & 4 & 3 & 2 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

make all elements of $\Delta = 0$

$R_2 - 3R_1$, $R_3 - 2R_1$, $R_4 - (R_1 + R_2 + R_3)$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 3$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 - R_3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \rho(A) = 3$$

Q.
$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 5 & 11 \end{bmatrix}$$

$R_1 \longleftrightarrow R_2$
$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 7 \\ 1 & 5 & 11 \end{bmatrix}$$

$R_3 - R_1 , \quad R_4 - R_1$
$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

$R_3 - 2R_2 , \quad R_4 - 4R_2$
$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$

System of linear Equations

Homogenous system

$AX = 0$ (Always consistent)

- Trivial solution — Zero sol^n
  - $\rho(A) =$ No. of variables or $|A| \neq 0$
- Non-Trivial solution — Non zero sol^n
  - $\rho(A) <$ No. of variables or $|A| = 0$

Non homogenous system

$AX = B \quad B \neq 0$

- Inconsistent
  - $\rho(A) \neq \rho(AB)$
- consistent
  - $\rho(A) = \rho(AB)$
  - unique sol^n: $\rho(A) = \rho(AB) =$ No. of variables or $|A| \neq 0$
  - ∞ sol^n: $\rho(A) = \rho(AB) <$ No. of variables

e.g; $x_1 + x_2 + 2x_3 + x_4 = 0$

$2x_1 + 2x_2 + 4x_3 + 2x_4 = 0$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

rank = 1, No. of variables = 4

Free variables = Total no. of variables − rank
$= 4 - 1 = 3$

The equation is $x_1 + x_2 + 2x_3 + x_4 = 0$ —— ①

let $x_1 = k_1$, $x_2 = k_2$, $x_3 = k_3$

$k_1 + k_2 + 2k_3 + x_4 = 0$

$x_4 = -k_1 - k_2 - 2k_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ -k_1-k_2-2k_3 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

Different values of k_1, k_2 & k_3 we can generate infinitely many solutions to the given system.

$\rho(A) <$ No. of variables ⇒ We have to introduce some free variables ⇒ ∞ many solutions.